ERTAN SİNAN ŞAHİN

CALCULUS 1

Lecture Notebook

This notebook is designed to be used alongside the video lessons. The QR code at the start of each chapter links to its video; the blank spaces on each page are filled in during the lesson.

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CHAPTER 1

LIMITS AND CONTINUITY

watch video

Rates of Change and Tangent Lines to Curves

Let us identify the two fundamental problems that gave rise to calculus: drawing a tangent line to a curve and computing the area under a curve. In this unit we will build the limit concept, which is the tool required by the first of these.

For an object released from rest, writing y for the distance fallen in meters and t for the elapsed time in seconds, the law of free fall is

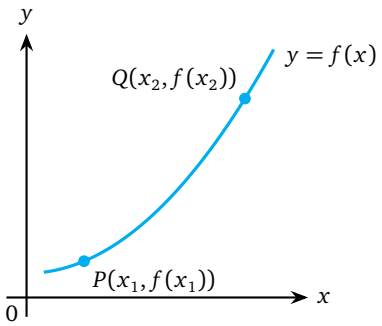
$$y = 4.9t^2$$

Let us discuss the difference between average speed and instantaneous speed by means of this law. Let us see why we are forced to take a limit in order to reach the instantaneous speed.

DEFINITION: The **average rate of change** of $y = f(x)$ over the interval $[x_1, x_2]$, where $h = x_2 - x_1 \neq 0$, is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}$$

Geometrically, this ratio is the slope of the **secant** line through the points P and Q on the curve.



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EXAMPLE 1: A bolt is dropped from the railing of a bridge. According to the law of free fall:

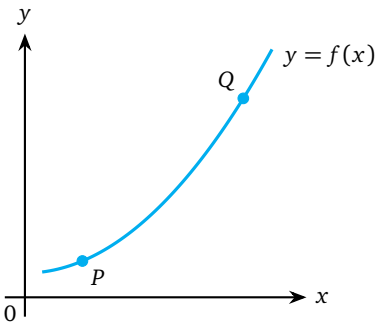
a. Find the average speed of the bolt during the first 4 s.

b. Write the average speed between $t = 3$ s and $t = 3 + h$ in terms of h.

c. Find the value this expression approaches as $h \to 0$, that is, the instantaneous speed at $t = 3$ s.

Tangent line to a curve

Let us examine what the slope of the secant line through the points P and Q approaches as Q moves along the curve toward P. Using this limit, let us define the slope of a curve at a point and the tangent line at that point.



EXAMPLE 2: Using the limit definition, find the slope of the tangent line to the curve $y = x^2$ at the point $P(1,1)$, and write the equation of the tangent line.

EXAMPLE 3: For $a \neq 0$, find the slope of the tangent line to the curve $y = \frac{1}{x}$ at the point with abscissa $x = a$. Determine the points at which this slope equals $-\frac{1}{4}$.

Limit of a Function and Limit Laws

Let us examine, through an example, how a function behaves as x approaches a point at which the function is not defined.

EXAMPLE 4: The function $f(x) = \frac{x^2+x}{x+1}$ is not defined at $x = -1$. Using a table, determine the number that the values of $f(x)$ approach as x approaches -1 from the left and from the right.

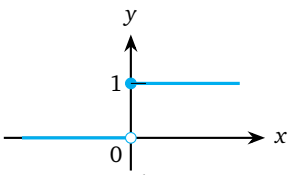
Let us compute the values together and write them in the table.

x	-1.1	-1.01	-1.001	-0.999	-0.99	-0.9
$f(x)$						

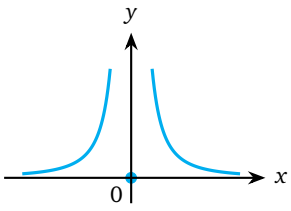
Starting from here, let us give the informal definition of the limit and write its notation. Let us emphasize that the limit of a function at a point is independent of its value at that point.

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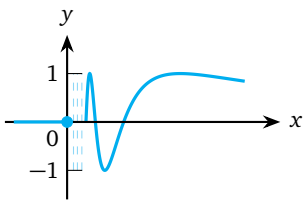
Let us discuss the three typical situations in which a limit fails to exist, using the graphs below: a jump, unbounded growth and infinite oscillation. In each case let us justify separately why the limit fails to exist as $x \to 0$.



$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$



$$g(x) = \begin{cases} \dfrac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$



$$f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\dfrac{1}{x}, & x > 0 \end{cases}$$

Let L, M, c and k be real numbers with $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then the following rules hold:

Sum Rule: $\lim_{x\to c}(f(x) + g(x)) = L + M$

Difference Rule: $\lim_{x\to c}(f(x) - g(x)) = L - M$

Constant Multiple Rule: $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$

Product Rule: $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$

Quotient Rule: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, for $M \neq 0$

Power Rule: $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer

Root Rule: $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

NOTE: In the root rule, if n is even we assume that $f(x) \geq 0$ on an interval containing c.

Let us see, through an example, how these rules work for polynomials, and for rational and radical expressions.

EXAMPLE 5: Using the limit laws, compute the following limits.

a. $\lim_{x\to 2}\left(x^3 - 2x^2 + 4x - 1\right)$

b. $\lim_{x\to 1} \frac{3x^2 - x + 2}{x^3 + 3}$

c. $\lim_{x\to 3} \sqrt{2x^2 + 7}$

Let us see, through an example, what we can do when, in the limit of a rational function as $x \to c$, substituting c for x makes both the numerator and the denominator zero.

EXAMPLE 6: Compute the limit $\lim_{x\to 3} \frac{x^2 - 2x - 3}{x^2 - 9}$.

Let us examine, again through an example, what to do when the same situation arises in an expression involving a radical.

EXAMPLE 7: Compute the limit $\lim_{x\to 0} \frac{\sqrt{x^2+9} - 3}{x^2}$.

EXAMPLE 8: Compute the limit $\lim_{x\to 0} \frac{\frac{1}{x+2} - \frac{1}{2}}{x}$.

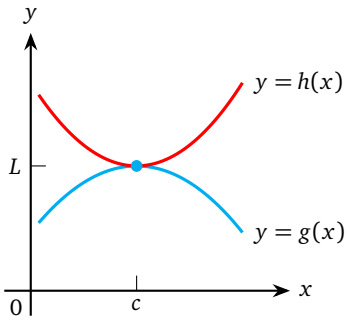
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Let us collect the three routes we take when substitution gives $\frac{0}{0}$.

Expression	Method
Factorable	Cancel
Radical	Conjugate
Compound fraction	Common denominator

The Sandwich Theorem

In the graph below, the curves g and h approach the same value L at $x = c$. Let us discuss on the graph why a function f squeezed between these two curves cannot approach any other value at $x = c$, and from this let us state the Sandwich Theorem.



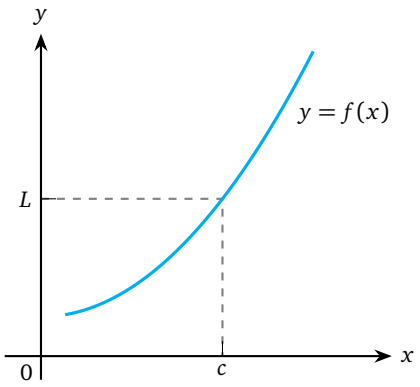
EXAMPLE 9: Find the following limits with the help of the Sandwich Theorem.

a. $\lim_{x\to 0} x^2 \cos\frac{3}{x}$

b. Given that $2-\frac{x^2}{3} \le u(x) \le 2+x^2$ for all x, what is $\lim_{x\to 0} u(x)$?

The Precise Definition of a Limit

Let us discuss why the phrase "sufficiently close" in the informal definition is inadequate. Let us ask how close x must be to c in order to bring the values of $f(x)$ within a prescribed tolerance ϵ of L.



DEFINITION: Let $f(x)$ be defined on an open interval containing c; it need not be defined at c itself. If, corresponding to every given number $\epsilon > 0$, there is a number $\delta > 0$ satisfying

$$0 < |x-c| < \delta \implies |f(x)-L| < \epsilon$$

then we say that the limit of $f(x)$ as x approaches c is L, and we write $\lim_{x\to c} f(x) = L$.

EXAMPLE 10: Using the precise definition of the limit, show that $\lim_{x\to 2}(3x-1) = 5$.

EXAMPLE 11: Let $f(x) = \sqrt{x}$, $c = 4$, $L = 2$ and $\epsilon = 0.1$. Find a value $\delta > 0$ satisfying the condition in the definition.

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EXAMPLE 12: Using the precise definition of the limit, prove that $\lim_{x\to 2} x^2 = 4$.

One-Sided Limits

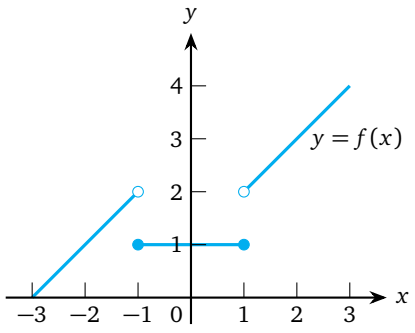
Let us define the behavior of a function as x approaches a point only from the right or only from the left. Let us write the notations for the right-hand limit ($x \to c^+$) and the left-hand limit ($x \to c^-$).

THEOREM: A function f has a limit at c if and only if it has both a left-hand and a right-hand limit there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

EXAMPLE 13: Find the left-hand and right-hand limits of the function $f(x) = \frac{|x-3|}{x-3}$ at $x = 3$. Does $\lim_{x\to 3} f(x)$ exist?

EXAMPLE 14: The graph below shows the function $y = f(x)$. Using this graph, determine the following values.



a. $\lim_{x\to -1^-} f(x)$

b. $\lim_{x\to -1^+} f(x)$

c. $\lim_{x\to -1} f(x)$

d. $f(-1)$

e. $\lim_{x\to 1^-} f(x)$

f. $\lim_{x\to 1^+} f(x)$

g. $\lim_{x\to 1} f(x)$

h. $f(1)$

THEOREM: With the angle measured in radians,

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

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EXAMPLE 15: Compute the following limits.

a. $\lim_{x\to 0} \frac{\sin 3x}{4x}$

b. $\lim_{x\to 0} \frac{\tan 3x}{\sin 5x}$

c. $\lim_{x\to 0} \frac{1-\cos x}{x}$

d. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

EXAMPLE 16: Compute the following limits.

a. $\lim_{x\to\infty} \left(3-\frac{2}{x}\right)$

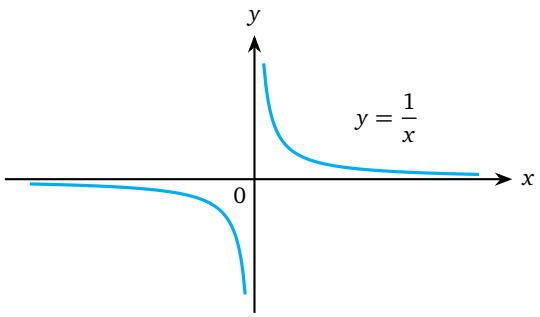
b. $\lim_{x\to\infty} \frac{7x^2-3x+4}{2x^2+5}$

c. $\lim_{x\to-\infty} \frac{6x+5}{x^3-2}$

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Limits Involving Infinity; Asymptotes of Graphs

Let us define the behavior of function values as x increases or decreases without bound. Then let us use this definition to show that $\lim_{x\to\infty} \frac{1}{x} = 0$, and note that $\lim_{x\to-\infty} \frac{1}{x} = 0$ is obtained in the same way.



DEFINITION: A line $y = b$ is a **horizontal asymptote** of the graph of $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

EXAMPLE 17: Find the horizontal asymptotes of the graph of $f(x) = \frac{3x-4}{|x|+2}$.

EXAMPLE 18: Compute the following limits.

a. $\lim_{x\to\pm\infty} x \sin\frac{1}{x}$ b. $\lim_{x\to 0^-} e^{1/x}$

EXAMPLE 19: Using the Sandwich Theorem, find the horizontal asymptote of the curve $y = 3 - \frac{\cos x}{x}$.

EXAMPLE 20: Find the horizontal asymptotes of the graph of $f(x) = \frac{\sqrt{4x^2+5}}{x-3}$.

EXAMPLE 21: Compute the limit $\lim_{x\to\infty}\left(x - \sqrt{x^2+25}\right)$.

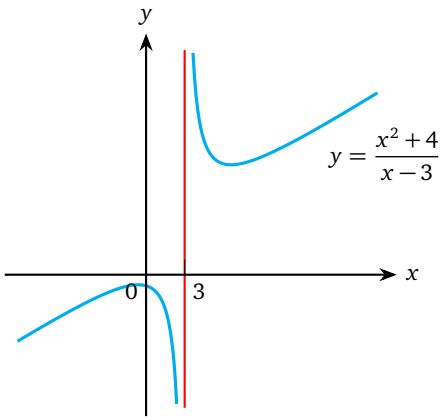
EXAMPLE 22: Compute the limit $\lim_{x\to\infty}\left(\sqrt{x^2+3x} - x\right)$.

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Oblique Asymptotes

Let us examine, through an example and by means of polynomial division, the concept of an **oblique asymptote**, which arises when the degree of the numerator is one more than that of the denominator.

EXAMPLE 23: Find the oblique asymptote of the graph of $f(x) = \frac{x^2+4}{x-3}$.



Infinite limits

Let us define the situation in which the values of a function grow without bound as x approaches a finite number, and write the notation $\lim_{x\to 0^+}\frac{1}{x} = \infty$. Let us emphasize that saying "the limit is infinity" does not mean the limit exists; it only describes the manner in which it fails to exist.

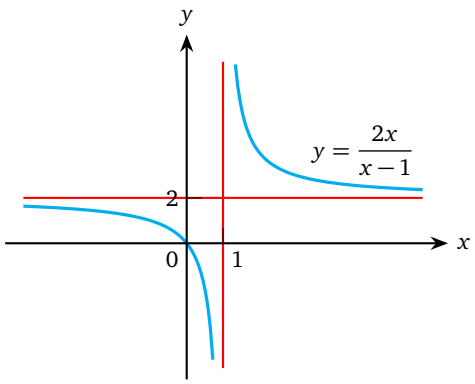
EXAMPLE 24: Compute the following limits.

a. $\lim_{x\to 0^-}\frac{1}{x}$ c. $\lim_{x\to 3^-}\frac{x+1}{x-3}$

b. $\lim_{x\to 2}\frac{1}{(x-2)^2}$ d. $\lim_{x\to 1}\frac{x+2}{x^2-1}$

DEFINITION: A line $x = a$ is a **vertical asymptote** of the graph of $y = f(x)$ if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

EXAMPLE 25: The graph of the function $f(x) = \frac{2x}{x-1}$ is given below. Using the graph, determine the following.



a. $\lim_{x\to 1^-} f(x)$

b. $\lim_{x\to 1^+} f(x)$

c. $\lim_{x\to\infty} f(x)$

d. $\lim_{x\to-\infty} f(x)$

e. Vertical asymptote

f. Horizontal asymptote

Dominant Terms

Let us examine, through an example, how a polynomial behaves like its highest-degree term alone when $|x|$ is large.

EXAMPLE 26: Let $f(x) = 2x^4 - 5x^3 + x - 7$ and $g(x) = 2x^4$. By computing $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)}$, show that the two functions become close to each other when $|x|$ is large.

EXAMPLE 27: Compute the following limit.

$$\lim_{x\to-\infty} \frac{x^2+3x-1}{x-2}$$

Let P and Q be polynomials. Let us collect the limit of the rational function $\frac{P}{Q}$ as $x \to \pm\infty$ according to the degrees; a and b are the leading coefficients of these polynomials.

Degrees	Limit
$\deg P < \deg Q$	0
$\deg P = \deg Q$	$\frac{a}{b}$
$\deg P > \deg Q$	$\pm\infty$

In the last row, if the degree difference is exactly 1, the graph also has an oblique asymptote.

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Continuity

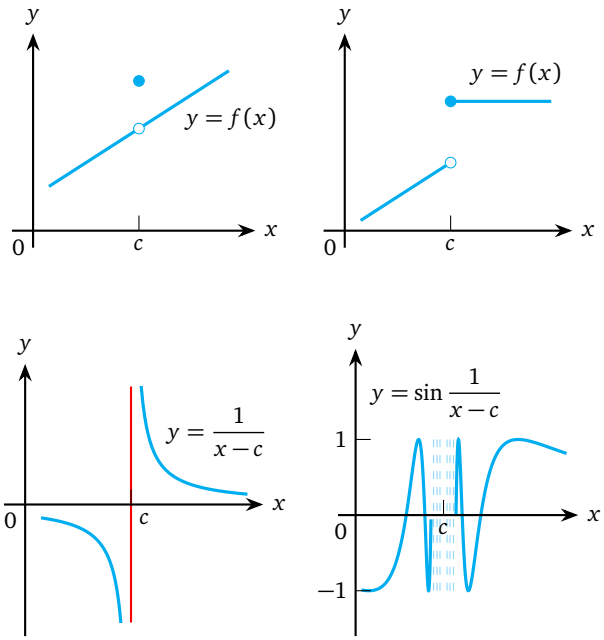
Starting from the idea that the graph of a function can be drawn "without lifting the pencil", let us give an intuitive introduction to the concept of continuity. Then let us examine, one by one, the three conditions required for a function to be continuous at a point.

DEFINITION: Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f. If

$$\lim_{x\to c} f(x) = f(c)$$

then f is said to be **continuous** at c. Similarly, f is **right-continuous** at c if $\lim_{x\to c^+} f(x) = f(c)$, and **left-continuous** at c if $\lim_{x\to c^-} f(x) = f(c)$. A function that is continuous at every point of its domain is called a **continuous function**.

In each of the four graphs below, the function f is discontinuous at $x = c$. Let us determine which condition of continuity fails in each case and name the type of the discontinuity.



EXAMPLE 28: For $x \neq 3$, let $g(x) = \frac{x^2-9}{x-3}$. What should the value $g(3)$ be so that the function is continuous at $x = 3$? What type of discontinuity is this?

EXAMPLE 29: Find the points at which the function $h(x) = \frac{x^2-x-2}{x^2-4}$ is discontinuous and state the type of each discontinuity.

EXAMPLE 30: Find the constants a and b so that the following function is continuous at every real number.

$$f(x) = \begin{cases} x^2 + a, & x < 2 \\ 3, & x = 2 \\ bx - 1, & x > 2 \end{cases}$$

Let f and g be continuous at c. Then the following functions are also continuous at c:

Sums: $f + g$

Differences: $f - g$

Constant multiples: $k \cdot f$, for any number k

Products: $f \cdot g$

Quotients: f/g, provided $g(c) \neq 0$

Powers: f^n, n a positive integer

Roots: $\sqrt[n]{f}$, n a positive integer, provided $\sqrt[n]{f}$ is defined on an interval containing c

Let us examine, through examples, the families of functions that we know to be continuous at every point of their domain.

Family of Functions	Example
Polynomial	$x^3 - 2x + 5$
Rational	$\frac{x+1}{x^2-4}$
Root	$\sqrt{x}$, $\sqrt[3]{x}$
Trigonometric	$\sin x$, $\cos x$
Exponential	e^x, 2^x
Logarithmic	$\ln x$, $\log_a x$

Let g be continuous at c and let f be continuous at $g(c)$. Let us discuss the continuity of the composite $f \circ g$ at c.

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This result turns into a tool for computing limits: if f is continuous, the limit symbol may be moved inside the function.

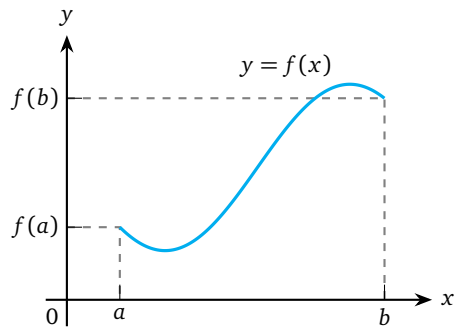
$$\lim_{x\to c} f\big(g(x)\big) = f\left(\lim_{x\to c} g(x)\right)$$

EXAMPLE 31: Compute the following limit.

$$\lim_{x\to 1}\sqrt{\frac{x^2-1}{x-1}}$$

The Intermediate Value Theorem

In the graph below, f is continuous on the closed interval $[a,b]$. Let us pick a value y_0 between $f(a)$ and $f(b)$ and examine on the graph whether the function attains this value on the interval.



Let us also discuss why continuity is indispensable here.

EXAMPLE 32: Show that the equation $x^3+x-3=0$ has at least one root in the interval $[1,2]$.

EXAMPLE 33: Using the Intermediate Value Theorem, show that the function $f(x)=x^3-12x+2$ has three distinct roots in the interval $[-4,4]$.

EXAMPLE 34: Use the Intermediate Value Theorem to show that the equation $\sqrt{3x+1}=5-x^2$ has a solution.

Exercises and Problems

1. The parabola $y=x^2-4x+3$ is given.

a. Find the average rate of change over the interval from $x=1$ to $x=4$.

b. Using the limit definition, write the equation of the tangent line at the point with abscissa $x=3$.

2. Compute the following limits.

a. $\lim_{x\to -2}\dfrac{x^3+8}{x^2-4}$

b. $\lim_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$

c. $\lim_{x\to 0}\dfrac{\frac{1}{2+x}-\frac{1}{2}}{x}$

d. $\lim_{x\to 0}\dfrac{\sin 5x}{\sin 8x}$

e. $\lim_{x\to 0}\dfrac{x^2}{1-\cos 2x}$

f. $\lim_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$

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3. It is known that $2x - x^2 \leq f(x) \leq 2\sqrt{x} - x$ for all $x \geq 0$. Find $\lim_{x\to 1} f(x)$ and state which theorem you used.

4. Using the precise definition of the limit, prove that $\lim_{x\to 3}(2x+1) = 7$. Write a suitable value of δ for the given $\epsilon = 0.02$.

5. Find the constants a and b so that the following function is continuous at every real number.

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x < 1 \\ ax + b, & 1 \leq x < 3 \\ 2x - 5, & x \geq 3 \end{cases}$$

6. The function $f(x) = \frac{2x^2-5x-3}{x^2-9}$ is given.

a. Find the points at which it is discontinuous.

b. State the type of each discontinuity.

c. Write its vertical asymptote, if any.

d. Write its horizontal asymptote, if any.

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7. Find the vertical and oblique asymptotes of the graph of $g(x) = \frac{2x^2-x+1}{x-1}$.

8. Show that there exists a number $c \in (0,1)$ with $f(c) = 0$ for the function $f(x) = x^3 - 3x + 1$.

Answers

Example questions

1 a. 19.6 m/s b. $29.4+4.9h$ m/s c. 29.4 m/s

2 $m=2,\ \ y=2x-1$

3 $m=-\frac{1}{a^2};\ \ \left(2,\frac{1}{2}\right),\left(-2,-\frac{1}{2}\right)$

4 -1

5 a. 7 b. 1 c. 5

6 $\frac{2}{3}$

7 $\frac{1}{6}$

8 $-\frac{1}{4}$

9 a. 0 b. 2

10 $\delta=\frac{\epsilon}{3}$

11 $\delta=0.39$

12 $\delta=\min\left\{1,\frac{\epsilon}{5}\right\}$

13 $-1;\ \ 1;\ \ \text{the limit does not exist}$

14 a. 2 b. 1 c. none d. 1
e. 1 f. 2 g. none h. 1

15 a. $\frac{3}{4}$ b. $\frac{3}{5}$ c. 0 d. $\frac{1}{2}$

16 a. 3 b. $\frac{7}{2}$ c. 0

17 $y=3$ and $y=-3$

18 a. 1 b. 0

19 $y=3$

20 $y=2$ and $y=-2$

21 0

22 $\frac{3}{2}$

23 $y=x+3$

24 a. $-\infty$ b. $+\infty$ c. $-\infty$ d. none

25 a. $-\infty$ b. $+\infty$ c. 2
d. 2 e. $x=1$ f. $y=2$

26 1

27 $-\infty$

28 $g(3)=6$; removable discontinuity

29 $x=2$: removable $x=-2$: infinite

30 $a=-1,\ \ b=2$

31 $\sqrt{2}$

32 $f(1)=-1<0<7=f(2)$

33 $(-4,-3),\ \ (0,1),\ \ (3,4)$

34 $f(0)=-4<0<\sqrt{7}-1=f(2)$

Exercises and problems

1 a. 1 b. $y=2x-6$

2 a. -3 b. $\frac{1}{4}$ c. $-\frac{1}{4}$ d. $\frac{5}{8}$ e. $\frac{1}{2}$ f. $2x$

3 1; Sandwich Theorem

4 $\delta=\frac{\epsilon}{2};\ \ \delta=0.01$

5 $a=-\frac{1}{2},\ \ b=\frac{5}{2}$

6 a. $x=3,\ x=-3$ b. removable, infinite
c. $x=-3$ d. $y=2$

7 $x=1;\ \ y=2x+1$

8 $f(0)=1>0>-1=f(1)$

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CHAPTER 2
DERIVATIVES

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Tangent Lines and the Derivative at a Point

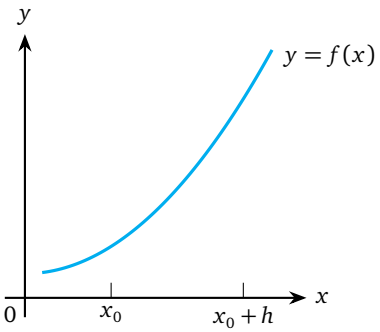
In Unit 1 we obtained the slope of a curve at a point as the limit of the slopes of secant lines. Let us give this limit a name and define it as the derivative of a function at a point.

DEFINITION: For $h \neq 0$, the expression

$$\frac{f(x_0+h)-f(x_0)}{h}$$

is called the **difference quotient** of f at x_0 with increment h. If this expression has a limit as $h \to 0$, this limit is called the **derivative** of f at x_0 and is denoted by $f'(x_0)$:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$



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EXAMPLE 1: Using the definition, find the derivative of the function $f(x) = x^3 + x$ at $x_0 = 1$. Write the equation of the tangent line to the curve at this point.

EXAMPLE 2: Using the definition, find the derivative of the function $f(x) = \sqrt{x}$ at $x_0 = 1$.

Let us examine, through an example, what the derivative measures when we read the difference quotient as an average rate of change.

EXAMPLE 3: A ball thrown vertically upward from the ground is at a height of $s(t) = 20t - 5t^2$ meters after t seconds. Using the definition of the derivative, find the instantaneous velocity of the ball at the following instants, and interpret the signs of the two results.

a. $t = 1$ s

b. $t = 3$ s

Let us collect the different interpretations of the limit of the difference quotient that we have met so far.

The limit below represents each of the following four quantities:

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

1. the slope of the graph of $y = f(x)$ at $x = x_0$,
2. the slope of the tangent line to the curve $y = f(x)$ at $x = x_0$,
3. the rate of change of $f(x)$ with respect to x at $x = x_0$,
4. the derivative $f'(x_0)$ of f at the point x_0.

Vertical tangent lines

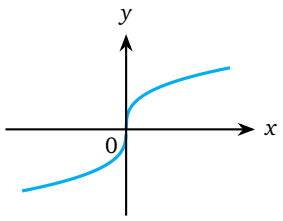
The limit of the difference quotient may grow without bound instead of being a finite number. Let us discuss what the tangent to the curve at that point should be in this case.

DEFINITION: For a continuous curve $y = f(x)$, if

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \infty \quad \text{or} \quad -\infty$$

then the curve has a **vertical tangent line** at the point where $x = x_0$.

Using this definition, let us show that the curve below has a vertical tangent line at the origin.



$y = x^{1/3}$

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The Derivative as a Function

So far we have computed the derivative at a single point x_0. Let us free this point and define the derivative as a new function that assigns a slope to each value of x.

DEFINITION: The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

(provided the limit exists). If $f'(x)$ exists, f is **differentiable** at x. Writing $z = x + h$, the same limit can be expressed equivalently as

$$f'(x) = \lim_{z\to x} \frac{f(z)-f(x)}{z-x}$$

EXAMPLE 4: Using the definition, find the derivative of the function $f(x) = x^3$. Then compute the values $f'(-1)$, $f'(0)$ and $f'(2)$.

EXAMPLE 5: Find the derivative of the function $f(x) = \sqrt{x}$ using the equivalent form of the derivative ($z \to x$). Find the slope of the tangent line to the curve at $x = 9$.

Let us examine the different notations used for the derivative and how the value of the derivative at a point is written.

NOTE: The following notations are used for the derivative of $y = f(x)$:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x)$$

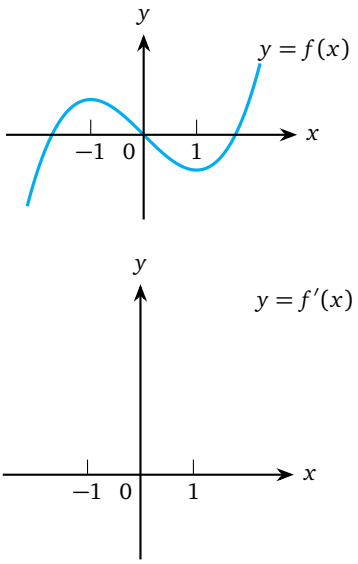
The value of the derivative at $x = a$ is written as

$$f'(a) = \frac{dy}{dx}\bigg|_{x=a} = \frac{d}{dx}f(x)\bigg|_{x=a}$$

Graphing the derivative

Let us see, through an example, how the graph of the derivative can be drawn by reading the slopes on the graph of a function.

EXAMPLE 6: The graph of the function $y = f(x)$ is given below. Using the slopes on the graph, sketch the graph of $y = f'(x)$ on the axes beneath it. At which points is f' zero, and on which interval is it negative?



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One-sided derivatives

Let us apply the one-sided limits of Unit 1 to the difference quotient and define what the derivative means at the endpoints of an interval.

DEFINITION: The **right-hand derivative** and the **left-hand derivative** of f at x_0 are, respectively, the limits

$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}, \qquad \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}$$

f is differentiable on the closed interval $[a, b]$ if it is differentiable at every point of (a, b) and has a right-hand derivative at a and a left-hand derivative at b. A function has a derivative at an interior point if and only if its right-hand and left-hand derivatives exist there and are equal.

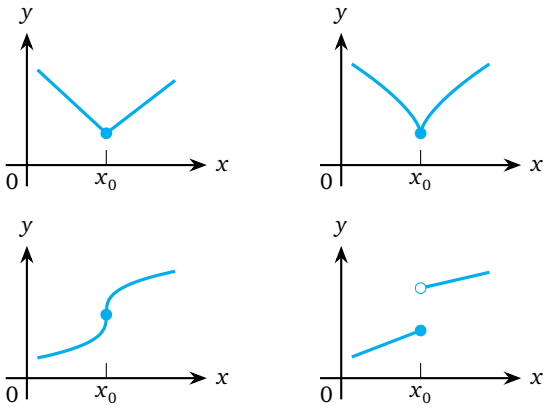
EXAMPLE 7: Find the right-hand and left-hand derivatives of the function $f(x) = |x|$ at $x = 0$. Is f differentiable at this point?

EXAMPLE 8: Determine whether the function below is differentiable at $x = 1$ by computing its right-hand and left-hand derivatives.

$$f(x) = \begin{cases} x^2, & x \le 1 \\ 2x - 1, & x > 1 \end{cases}$$

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Let us discuss, using the graphs below, the typical situations in which a function fails to have a derivative at a point. In each case, let us examine what happens to the slopes of the secant lines through the point $P(x_0, f(x_0))$.



Let us state and prove the relationship between differentiability and continuity.

THEOREM: If f has a derivative at $x = c$, then f is continuous at $x = c$.

EXAMPLE 9: Determine whether the function below is differentiable at $x = 1$, and justify your answer.

$$f(x) = \begin{cases} x + 1, & x < 1 \\ x^2, & x \ge 1 \end{cases}$$

Differentiation Rules

So far we have computed every derivative as the limit of the difference quotient. For polynomials, roots and exponential functions, let us do this once and write the result as a rule.

Let us first derive the derivatives of constant functions and powers of x.

THEOREM: Let c be a constant and n any real number. Then

$$\frac{d}{dx}(c) = 0, \qquad \frac{d}{dx}x^n = nx^{n-1}$$

hold for every x at which the powers are defined.

EXAMPLE 10: Find the derivatives of the following functions using the power rule.

a. x^5

b. $\dfrac{1}{x^3}$

c. $x^{3/2}$

d. $\sqrt[3]{x^5}$

Let us examine how the derivative of a constant multiple of a function and of a sum of two functions is found.

THEOREM: Let u and v be differentiable functions and c a constant. Then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}, \qquad \frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

hold.

EXAMPLE 11: Find the derivative of the following function.

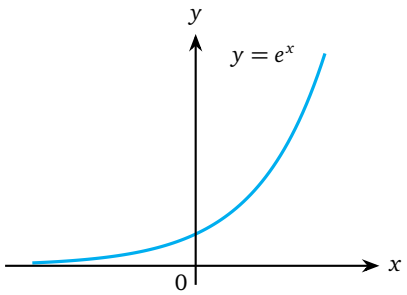
$$y = \frac{x^4}{4} - 2x^3 + 7x - 3$$

The derivative of the exponential function

Let us derive the derivative of the exponential function e^x from the difference quotient.

THEOREM: The derivative of the exponential function e^x is equal to itself:

$$\frac{d}{dx}e^x = e^x$$



EXAMPLE 12: Find the derivative of $f(x) = 2e^x - 5x^2 + 3$ and compute $f'(0)$.

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Product and quotient

Let us examine, through an example, whether the derivative of the product of two functions equals the product of their derivatives.

EXAMPLE 13: Let $f(x) = x^2$ and $g(x) = x^3$. Compute $(fg)'(x)$ and $f'(x)g'(x)$ and compare them.

Let us write the rules that give the derivative of a product and of a quotient.

THEOREM: If u and v are differentiable functions, then

$$\frac{d}{dx}(uv) = u'v + uv'$$

This is called the **product rule**.

THEOREM: If u and v are differentiable functions and $v \neq 0$, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

This is called the **quotient rule**.

EXAMPLE 14: Find the derivative of $y = (x^3 - 2x)(x^2 + 4)$ using the product rule.

EXAMPLE 15: Find the derivative of the following function using the quotient rule.

$$y = \frac{x^2}{x+3}$$

Higher-order derivatives

Let us define higher-order derivatives by differentiating the derivative, and write their notations.

DEFINITION: The derivative of the derivative of $y = f(x)$ is called the **second derivative**. If differentiation is applied n times, we obtain the **nth-order derivative**:

$$y'' = \frac{d^2y}{dx^2} = f''(x), \qquad y^{(n)} = \frac{d^ny}{dx^n} = f^{(n)}(x)$$

EXAMPLE 16: Find the first four derivatives of the following function.

$$f(x) = \frac{1}{x}$$

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Derivatives and Rates of Change

Let us interpret, through an example, the derivative as the instantaneous rate of change of one quantity with respect to another, and see how to read the unit of the derivative.

EXAMPLE 17: The surface area of a sphere is given by $S = 4\pi r^2$. Find the rate of change of the surface area with respect to the radius, with its unit, when the radius is $r = 3$ m.

Motion along a line

Let us define velocity, speed and acceleration from the position of a body moving along a line.

DEFINITION: Let $s = f(t)$ be the position of a body on the s-axis. The **displacement** between t and $t + \Delta t$ is $\Delta s = f(t + \Delta t) - f(t)$, the **average velocity** is $\dfrac{\Delta s}{\Delta t}$, and the **velocity** at time t is defined by

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

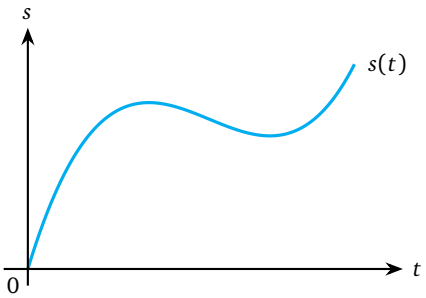
where the velocity is the instantaneous rate of change of the position.

DEFINITION: The absolute value of velocity is called **speed**, the derivative of velocity with respect to time is called **acceleration**, and the derivative of acceleration with respect to time is called **jerk**:

$$|v(t)| = \left|\frac{ds}{dt}\right|, \qquad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

EXAMPLE 18: The position of a body moving along a line, t seconds from the start, is $s(t) = t^3 - 9t^2 + 24t$ meters $(t \geq 0)$.



a. Find $v(t)$ and $a(t)$.

b. Find the times at which the body is at rest.

c. At each of the times $t = 1$ and $t = 5$, determine from the signs of the velocity and the acceleration whether the speed of the body is increasing or decreasing.

Derivatives in economics

In economics, a derivative is called marginal. Let us examine what the derivative of the production cost means.

NOTE: If the cost of producing x units is $C(x)$, the derivative $C'(x)$ is called the **marginal cost**. It is approximately equal to the additional cost of producing one more unit:

$$C'(x) \approx C(x+1) - C(x)$$

EXAMPLE 19: The cost of producing x items per day in a workshop is $C(x) = 200 + 10x + x^2$ lira.

a. Find the marginal cost $C'(x)$.

b. Find the marginal cost when $x = 20$ and compare it with $C(21) - C(20)$, the actual additional cost of the 21st item.

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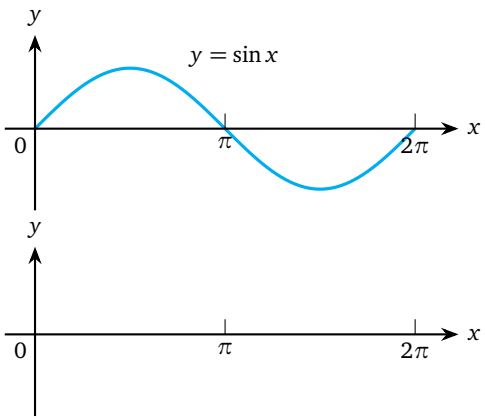
Derivatives of Trigonometric Functions

Let us find the derivatives of $\sin x$ and $\cos x$ using the two trigonometric limits of Unit 1.

THEOREM: Let x be measured in radians. Then

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x$$

hold.



EXAMPLE 20: Find the derivatives of the following functions.

a. $y = 2x^3 + 4\cos x - 7$ c. $y = \dfrac{\cos x}{x^2}$

b. $y = x^3 \sin x$

Simple harmonic motion

Let us examine, through an example, how the acceleration of a mass attached to a spring is related to its position.

EXAMPLE 21: A mass attached to a spring oscillates about its equilibrium position. Its position t seconds from the start is $s(t) = 8\sin t$ centimeters.

a. Find $v(t)$ and $a(t)$.

b. Find the position, velocity and acceleration of the mass at $t = \pi/2$.

The other four trigonometric functions

Let us derive the derivatives of tangent, cotangent, secant and cosecant from the derivatives of sine and cosine.

THEOREM: For every x at which they are defined,

$$\frac{d}{dx}(\tan x) = \sec^2 x, \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x, \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

hold.

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EXAMPLE 22: Find the derivatives of the following functions.

a. $y = 3\sec x - 2\cot x$ b. $y = e^x \sec x$

The Chain Rule

With the rules so far, we cannot differentiate composite functions such as $\sin(x^2 - 4)$. Let us derive a new rule for this.

Let us examine, through an example, how the derivatives combine in a composite function.

EXAMPLE 23: Let $y = 4u$ and $u = 3x$. Find the following derivatives and examine the relationship among them.

$$\frac{dy}{du}, \quad \frac{du}{dx}, \quad \frac{dy}{dx}$$

Let us write the rule that gives the derivative of a composite function.

THEOREM: If $y = f(u)$ and $u = g(x)$ are differentiable functions, then the composite $y = f(g(x))$ is also differentiable and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

holds. The same rule can also be written as:

$$(f \circ g)'(x) = f'\big(g(x)\big) \cdot g'(x)$$

EXAMPLE 24: Find the derivatives of the following functions using the chain rule.

a. $y = e^{x^2}$

b. $y = \cos(5x + 2)$

c. $y = \tan\left(x^2\right)$

Repeated use of the chain rule

Let us see, through an example, how we apply the chain rule in succession when more than two functions are nested.

EXAMPLE 25: Find the derivative of $y = \cos\left(e^{3x}\right)$.

A power of a function

Let us write a special form of the chain rule for the derivative of a power of a function.

THEOREM: If u is a differentiable function of x and n is a real number, then

$$\frac{d}{dx}(u^n) = n\,u^{n-1}\,\frac{du}{dx}$$

holds.

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EXAMPLE 26: Find the derivatives of the following functions.

a. $y = (x^3 - 2x)^5$

c. $y = \sqrt{x^2 + 9}$

b. $y = \dfrac{1}{(4x+3)^2}$

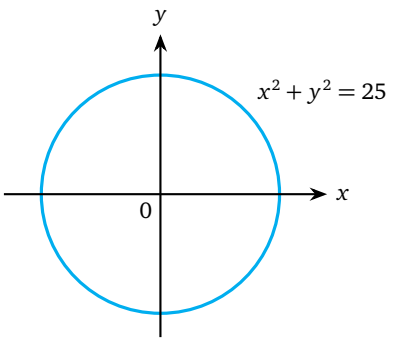
EXAMPLE 27: Find the derivative of the following function.

$$y = x\, e^{-x^2}$$

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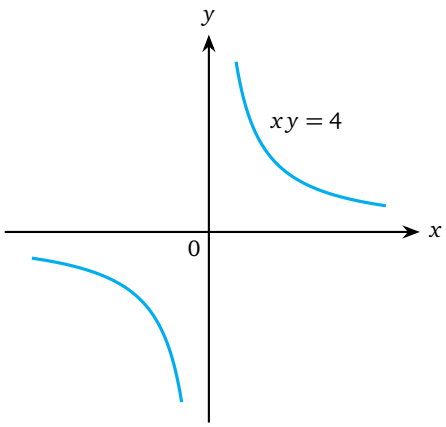
Implicit Differentiation

In some equations we cannot write y explicitly in terms of x, or the equation does not define a single function. Let us examine how to find the derivative in such cases.



Let us find the same derivative through an example, first by solving for y and then without solving.

EXAMPLE 28: Find the derivative y' for the equation $xy = 4$ in two ways: (i) by solving for y in terms of x, (ii) by differentiating both sides of the equation with respect to x.



Let us write how to treat the terms containing y when we differentiate both sides of an equation.

> **NOTE:** In implicit differentiation, y is regarded as a function of x; terms containing y are differentiated by the chain rule. For example,
>
> $$\frac{d}{dx}(y^2) = 2y\,\frac{dy}{dx}$$

EXAMPLE 29: Find the derivative y' for y defined implicitly by the following equations.

a. $y^3 + xy = 8$

b. $x^2 + y^2 = e^x$

The second derivative

Let us see, through an example, how the second derivative of an implicitly defined function is found.

EXAMPLE 30: Find the derivatives y' and y'' for y defined implicitly by the equation $x^2 - y^2 = 4$.

Tangent and normal lines

Let us see, through an example, how the tangent and normal lines of an implicitly defined curve at a point are found.

EXAMPLE 31: Show that the point $(1, 1)$ lies on the curve $x^2y + y^3 = 2$.

a. Find the equation of the tangent line to the curve at this point.

b. Find the equation of the normal line to the curve at this point.

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Derivatives of Inverse and Logarithmic Functions

Let us examine the relationship between the slopes of a function and its inverse.

THEOREM: Let f be a one-to-one differentiable function. If $f'(f^{-1}(b)) \neq 0$, then f^{-1} is differentiable at b and

$$(f^{-1})'(b) = \frac{1}{f'\left(f^{-1}(b)\right)}$$

holds.

EXAMPLE 32: The function $f(x) = 2x^3 + x$ is one-to-one. Find the value of $(f^{-1})'(3)$.

The derivative of the natural logarithm

Let us find the derivative of $\ln x$, the inverse of the exponential function.

THEOREM: For $x > 0$,

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

holds. If u is a differentiable and positive function of x, then the chain rule gives

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

This is the chain-rule form.

EXAMPLE 33: Find the derivatives of the following functions.

a. $y = \ln\left(3x^2 + 1\right)$ c. $y = \ln(\sin x)$

b. $y = x \ln x$

The derivatives of a^x and $\log_a x$

Let us find the derivatives of exponential and logarithmic functions whose base is not e.

THEOREM: For $a > 0$, $a \neq 1$,

$$\frac{d}{dx}(a^x) = a^x \ln a, \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

hold.

EXAMPLE 34: Find the derivatives of the following functions.

a. $y = 2^{x^2}$ c. $y = x \cdot 3^x$

b. $y = \log_5(x^2 + 1)$

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Logarithmic differentiation

Let us see, through an example, how to differentiate functions with many factors, quotients and powers by taking the logarithm of both sides.

EXAMPLE 35: For $x > 3$, find the derivative of the following function using logarithmic differentiation.

$$y = \frac{x^2\sqrt{x+1}}{(x-3)^3}$$

EXAMPLE 36: For $x > 0$, find the derivative of $y = x^{\sin x}$.

The general power rule

Let us prove, for every real exponent, the power rule that we accepted in the section on Differentiation Rules.

DEFINITION: For $x > 0$ and any real number n, the nth power of x is defined by

$$x^n = e^{n \ln x}$$

This definition holds for every real exponent n.

Let us write the number e as a limit using the derivatives of the exponential function and the logarithm.

THEOREM: The number e can be written as a limit:

$$e = \lim_{x \to 0} (1+x)^{1/x}$$

This limit is used to compute e.

Derivatives of Inverse Trigonometric Functions

Let us derive the derivatives of arcsin, arctan and arcsec from the inverse function derivative rule.

THEOREM: For every x at which they are defined,

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}} \quad (|x| > 1)$$

hold.

EXAMPLE 37: Find the derivatives of the following functions.

a. $y = \arcsin(5x)$

b. $y = \arctan(e^x)$

c. $y = \operatorname{arcsec}(x^2),\ x > 1$

EXAMPLE 38: Find the derivatives of the following functions.

a. $y = \arctan\sqrt{x}$

b. $y = x \arcsin x$

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The other three inverse trigonometric functions

Let us derive the derivatives of arccos, arccot and arccsc from the inverse cofunction identities.

THEOREM: Inverse cofunction identities

$$\arccos x = \frac{\pi}{2} - \arcsin x, \qquad \operatorname{arccot} x = \frac{\pi}{2} - \arctan x$$

$$\operatorname{arccsc} x = \frac{\pi}{2} - \operatorname{arcsec} x$$

Since these identities hold, the derivatives of the functions on the left are the negatives of the derivatives of the corresponding functions on the right:

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Related Rates

Let us see, through an example, how the rates of change of two interrelated quantities are related.

EXAMPLE 39: A spherical balloon is being inflated at a constant rate of 100π cm^3 per minute. Find the rate at which the radius is increasing when the radius is 5 cm.

Let us write the steps for solving such problems.

> **NOTE:** In related rates problems the following steps are followed: (1) Draw a figure and name the variables. (2) Write the relationship among the variables. (3) Differentiate both sides of the relationship with respect to t (chain rule). (4) Substitute the known values **after differentiating** and find the required rate.

EXAMPLE 40: A person 1.8 m tall walks away from the pole of a lamp 6 m above the ground at 1.5 m/s.

a. How fast is the tip of the shadow moving?

b. How fast is the length of the shadow increasing?

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EXAMPLE 41: The longer side of a rectangle is increasing at 3 cm/s while its shorter side is decreasing at 2 cm/s. When the longer side is 10 cm and the shorter side is 6 cm, find the rates of change of the following.

a. The area of the rectangle

b. The perimeter of the rectangle

EXAMPLE 42: A point moves on the circle $x^2 + y^2 = 25$. When the point is at $(3, 4)$, its x-coordinate is increasing at 2 units per second. At what rate is its y-coordinate changing at that instant?

Linearization and Differentials

Let us see, through an example, how we can approximate the value of a function near a point using its tangent line.

EXAMPLE 43: Write the equation of the tangent line to $f(x) = 1/x$ at $x = 2$ as $L(x)$. Using this line, approximate $1/2.05$.

Let us write this approximation as a definition.

DEFINITION: If f is differentiable at $x = a$, then

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a. The point a is called the **center** of the approximation, and for x near a,

$$f(x) \approx L(x)$$

holds.

EXAMPLE 44: Find the linearization of $f(x) = \sqrt{x}$ at $x = 4$ and use it to approximate $\sqrt{4.1}$.

Differentials

Let us use the derivative to estimate the change in y caused by a small change in x.

DEFINITION: Let $y = f(x)$ be a differentiable function. Here dx is an independent variable, and the **differential** of y is defined by

$$dy = f'(x)\,dx$$

When $dx = \Delta x$ is small, $\Delta y \approx dy$.

EXAMPLE 45: Find the differential dy of $y = x \ln x$. Compute dy when $x = 1$ and $dx = 0.2$.

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Estimation with differentials

Let us see, through an example, how to approximate a value using differentials.

EXAMPLE 46: Use $f(x) = \sqrt{x}$ with center $a = 25$ and $dx = 1$ to approximate $\sqrt{26}$.

EXAMPLE 47: The height of a cylinder is measured as 20 cm and its radius as 5 cm. The error in the radius measurement is at most 0.2 cm. Using differentials, find the estimated error and the percentage error in the computed volume.

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Exercises and Problems

1. Find the derivative of the following function using the definition of the derivative. Then write the equation of its tangent line at $x = 2$.

$$f(x) = \frac{3}{x+1}$$

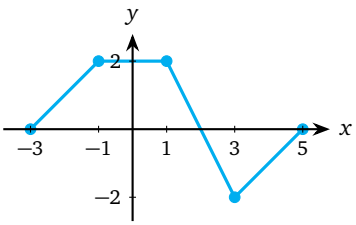
2. Find the values of the constants a and b for which the following function is differentiable at $x = 1$.

$$f(x) = \begin{cases} x^3, & x \leq 1 \\ ax + b, & x > 1 \end{cases}$$

3. The graph of $y = f(x)$ is given below. Using the graph, find the following.

a. Find $f'(-2)$, $f'(0)$, $f'(2)$ and $f'(4)$.

b. Find the points at which f is not differentiable.



4. Find the derivatives of the following functions.

a. $y = \sin^3(2x)$

b. $y = \ln\left(\dfrac{x+1}{x-1}\right)$, $x > 1$

5. Find the points at which the curve $x^2 - xy + y^2 = 3$ has horizontal tangent lines.

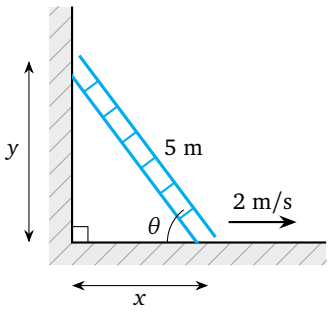
6. For $x > 0$, find the derivative of $y = x^{x^2}$ and compute $y'(1)$.

7. For $x > 0$, find the derivative of $f(x) = \arctan x + \arctan(1/x)$.

8. A 5 m ladder leans against a vertical wall. Its lower end is sliding along the ground away from the wall at 2 m/s. When the lower end is 3 m from the wall, find the following.

a. How fast is the top of the ladder sliding down the wall?

b. How fast is the angle θ between the ladder and the ground changing?



9. Find the linearization of $f(x) = e^x$ at $x = 0$ and use it to approximate $e^{0.05}$.

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Answers

Example questions

1 $f'(1)=4,\quad y=4x-2$

2 $f'(1)=\frac{1}{2}$

3 a. 10 m/s (rising) b. −10 m/s (falling)

4 $f'(x)=3x^2;\quad 3,\quad 0,\quad 12$

5 $f'(x)=\frac{1}{2\sqrt{x}},\quad f'(9)=\frac{1}{6}$

6 $f'(\pm1)=0;\quad f'<0$ on $(-1,1)$

7 $1,\quad -1;$ not differentiable

8 differentiable, $f'(1)=2$

9 Not differentiable; f is discontinuous at $x=1$.

10 a. $5x^4$ b. $-\frac{3}{x^4}$ c. $\frac{3}{2}\sqrt{x}$ d. $\frac{5}{3}\sqrt[3]{x^2}$

11 $\frac{dy}{dx}=x^3-6x^2+7$

12 $f'(x)=2e^x-10x;\quad f'(0)=2$

13 $(fg)'(x)=5x^4,\quad f'(x)g'(x)=6x^3;$ not equal

14 $y'=5x^4+6x^2-8$

15 $y'=\frac{x(x+6)}{(x+3)^2}$

16 $-\frac{1}{x^2},\quad \frac{2}{x^3},\quad -\frac{6}{x^4},\quad \frac{24}{x^5}$

17 $\frac{dS}{dr}=8\pi r;\quad 24\pi$ m^2/m

18
a. $v(t)=3t^2-18t+24,\quad a(t)=6t-18$
b. $t=2$ and $t=4$
c. $t=1$: slowing down; $t=5$: speeding up

19
a. $C'(x)=10+2x$
b. $C'(20)=50$ lira, $C(21)-C(20)=51$ lira

20
a. $6x^2-4\sin x$ b. $3x^2\sin x+x^3\cos x$
c. $-\frac{x\sin x+2\cos x}{x^3}$

21
a. $v(t)=8\cos t,\quad a(t)=-8\sin t$
b. $s=8$ cm, $v=0$, $a=-8$ cm/s^2

22 a. $3\sec x\tan x+2\csc^2 x$ b. $e^x\sec x\,(1+\tan x)$

23 $\frac{dy}{du}=4,\quad \frac{du}{dx}=3,\quad \frac{dy}{dx}=12=4\cdot3$

24 a. $2x\,e^{x^2}$ b. $-5\sin(5x+2)$ c. $2x\sec^2(x^2)$

25 $y'=-3e^{3x}\sin\left(e^{3x}\right)$

26 a. $5(x^3-2x)^4(3x^2-2)$ b. $-\frac{8}{(4x+3)^3}$ c. $\frac{x}{\sqrt{x^2+9}}$

27 $y'=(1-2x^2)\,e^{-x^2}$

28 i. $y'=-\frac{4}{x^2}$ ii. $y'=-\frac{y}{x}$

29 a. $-\frac{y}{3y^2+x}$ b. $\frac{e^x-2x}{2y}$

30 $y'=\frac{x}{y},\quad y''=-\frac{4}{y^3}$

31 a. $y=-\frac{1}{2}x+\frac{3}{2}$ b. $y=2x-1$

32 $\frac{1}{7}$

33 a. $\frac{6x}{3x^2+1}$ b. $\ln x+1$ c. $\cot x$

34 a. $2x\cdot2^{x^2}\ln2$ b. $\frac{2x}{(x^2+1)\ln5}$ c. $3^x\,(1+x\ln3)$

35 $y'=\frac{x^2\sqrt{x+1}}{(x-3)^3}\left(\frac{2}{x}+\frac{1}{2(x+1)}-\frac{3}{x-3}\right)$

36 $y'=x^{\sin x}\left(\cos x\ln x+\frac{\sin x}{x}\right)$

37 a. $\frac{5}{\sqrt{1-25x^2}}$ b. $\frac{e^x}{1+e^{2x}}$ c. $\frac{2}{x\sqrt{x^4-1}}$

38 a. $\frac{1}{2\sqrt{x}(1+x)}$ b. $\arcsin x+\frac{x}{\sqrt{1-x^2}}$

39 $\frac{dr}{dt}=1$ cm/min

40 a. $\frac{15}{7}$ m/s b. $\frac{9}{14}$ m/s

41 a. -2 cm^2/s b. 2 cm/s

42 $\frac{dy}{dt}=-\frac{3}{2}$ units/s (decreasing)

43 $L(x)=\frac{1}{2}-\frac{x-2}{4};\quad L(2.05)=0.4875$

44 $L(x)=\frac{x}{4}+1;\quad \sqrt{4.1}\approx2.025$

45 $dy=(\ln x+1)\,dx;\quad dy=0.2$

46 $\sqrt{26}\approx5.1$

47 40π cm^3, %8

Exercises and problems

1 $f'(x)=-\frac{3}{(x+1)^2}\qquad y=-\frac{x}{3}+\frac{5}{3}$

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2 $a = 3, \quad b = -2$

3 a. $1, \quad 0, \quad -2, \quad 1$ b. $x = -1, \quad x = 1, \quad x = 3$

4 a. $6\sin^2(2x)\cos(2x)$ b. $-\frac{2}{x^2-1}$

5 $(1,2)$ and $(-1,-2)$

6 $y' = x^{x^2}(2x\ln x + x); \quad y'(1) = 1$

7 $f'(x) = 0$

8 a. $\frac{3}{2}$ m/s b. $-\frac{1}{2}$ rad/s

9 $L(x) = 1 + x; \quad e^{0.05} \approx 1.05$

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CHAPTER 3

APPLICATIONS OF DERIVATIVES

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Extreme Values of Functions on Closed Intervals

Let us examine how to find the greatest and least values a function takes on an interval using derivatives. First, let us define these values.

DEFINITION: Let f be a function with domain D and let $c \in D$. If $f(x) \leq f(c)$ for every $x \in D$, then f has an **absolute maximum** value on D at c; if $f(x) \geq f(c)$ for every $x \in D$, then f has an **absolute minimum** value on D at c. The maximum and minimum values are called the **extreme values** of f.

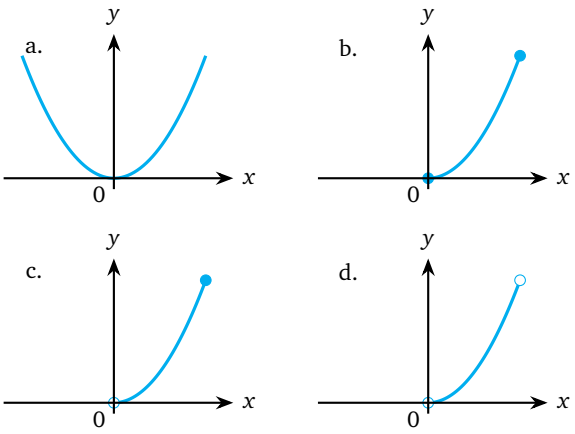
EXAMPLE 1: The graph of $y = x^2$ is shown below on four different domains. For each one, find the absolute maximum and absolute minimum values of the function, if they exist.

a. $(-\infty, \infty)$

b. $[0, 3]$

c. $(0, 3]$

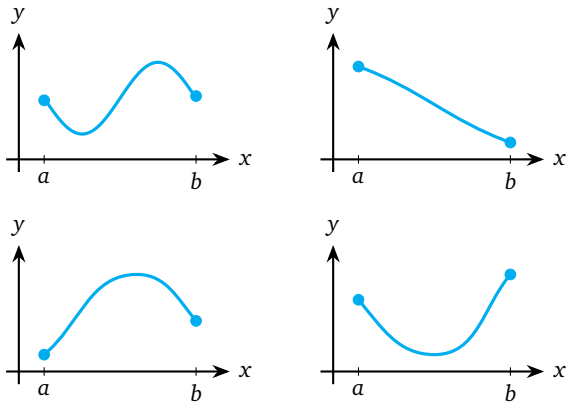
d. $(0, 3)$



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Let us state the extreme value theorem, which says that a continuous function on a closed interval attains its extreme values.

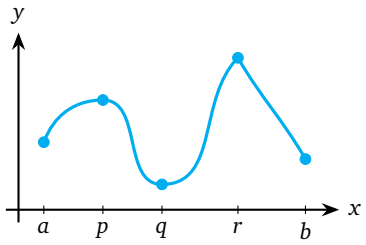
THEOREM: If f is continuous on the closed interval $[a, b]$, then f attains an absolute maximum value M and an absolute minimum value m on this interval; that is, there exist $x_1, x_2 \in [a, b]$ with $f(x_1) = m$ and $f(x_2) = M$, and $m \leq f(x) \leq M$ for every $x \in [a, b]$.



Let us also define what it means for a function to take its greatest or least value only near a point.

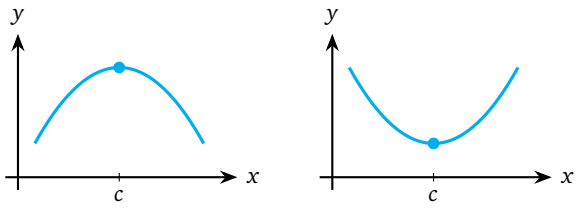
DEFINITION: Let f be a function with domain D and let $c \in D$. If $f(x) \leq f(c)$ for every $x \in D$ in some open interval containing c, then f has a **local maximum** value at c; if $f(x) \geq f(c)$ for every $x \in D$ in such an interval, then f has a **local minimum** value at c.

EXAMPLE 2: The graph of a function f that is defined and continuous on the closed interval $[a, b]$ is given below. Identify the points where the absolute maximum, absolute minimum, local maximum and local minimum values are attained.



Let us examine what must happen to the derivative at an interior point where a function takes a local extreme value.

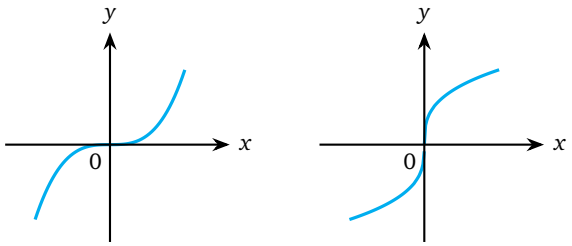
THEOREM: If f takes a local maximum or local minimum value at an interior point c of its domain and $f'(c)$ exists, then $f'(c) = 0$.



Let us give a name to the points where a local extreme value can occur.

DEFINITION: An interior point c of the domain of f where $f'(c) = 0$ or $f'(c)$ does not exist is called a **critical point** of f.

Let us discuss, using the two graphs below, whether a local extreme value is attained at every critical point.



$y = x^3$ $y = x^{1/3}$

EXAMPLE 3: Find the critical points of the following functions.

a. $f(x) = x^4 - 4x^3$ c. $h(x) = \sqrt[3]{x^2 - 1}$

b. $g(x) = x^2 e^{-x}$

Let us write down the method for finding the absolute extreme values of a continuous function on a closed interval.

NOTE: If f is continuous on $[a, b]$, its absolute extreme values are found in the following steps:

1. Find the critical points in (a, b).
2. Evaluate f at the critical points and at the endpoints a and b.
3. The largest of these values is the absolute maximum and the smallest is the absolute minimum.

EXAMPLE 4: Find the absolute extreme values of $f(x) = x^3 - 6x^2 + 9x - 1$ on the interval $[0, 4]$ and the points where they are attained.

EXAMPLE 5: Find the absolute extreme values of $f(x) = x - 2\ln x$ on the interval $[1, 4]$ and the points where they are attained.

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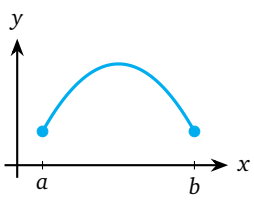
EXAMPLE 6: Find the absolute extreme values of $f(x) = x + 2\cos x$ on the interval $[0, \pi]$ and the points where they are attained.

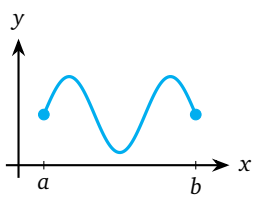
EXAMPLE 7: Find the absolute and local extreme values of $f(x) = |x^2 - 4|$ on the interval $[-3, 3]$, together with the points where they are attained.

The Mean Value Theorem

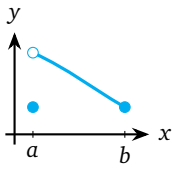
Let us first state Rolle's theorem.

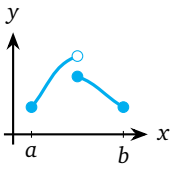
THEOREM: Let $y = f(x)$ be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number $c \in (a, b)$ such that $f'(c) = 0$.

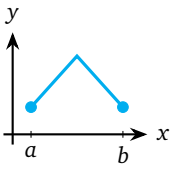




Let us discuss, using the three graphs below, how a horizontal tangent may fail to exist when one of the hypotheses of Rolle's theorem is not satisfied.







EXAMPLE 8: Show that the equation $2x - \cos x = 0$ has exactly one real solution.

EXAMPLE 9: Show that $f(x) = x^3 - 4x$ satisfies the hypotheses of Rolle's theorem on the interval $[-2, 2]$ and find the values of c given by the conclusion of the theorem.

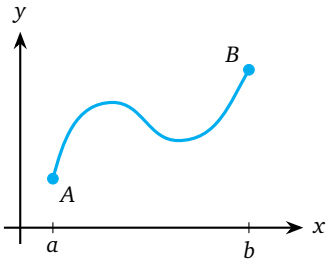
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Let us state the mean value theorem, a generalization of Rolle's theorem.

THEOREM: If $y = f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

for at least one number $c \in (a, b)$.



EXAMPLE 10: Find the value of c given by the conclusion of the mean value theorem for $f(x) = x^3$ on the interval $[1, 3]$.

Let us examine, through an example, the relationship between the average rate of change and the instantaneous rate of change.

EXAMPLE 11: A plant grows from 5 cm to 45 cm in 20 days. Assuming that the height of the plant is a continuous and differentiable function of time, show that at some moment during this period the growth rate is exactly 2 cm/day.

Consequences of the Mean Value Theorem

Let us state two consequences of the mean value theorem.

THEOREM: If $f'(x) = 0$ for every $x \in (a, b)$, then there is a constant C such that $f(x) = C$ on (a, b).

THEOREM: If $f'(x) = g'(x)$ for every $x \in (a, b)$, then there is a constant C such that $f(x) = g(x) + C$ on (a, b).

EXAMPLE 12: Show that $f(x) = \arctan x + \arctan \frac{1}{x}$ is constant for $x > 0$ and find this constant.

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EXAMPLE 13: Find the function f whose derivative is $f'(x) = \frac{1}{x} + 2x$ $(x > 0)$ and whose graph passes through the point $(1, 3)$.

Let us see, through an example, how we find the velocity and position functions of an object from its acceleration.

EXAMPLE 14: The acceleration of an object moving along a straight line is $a(t) = (6t - 4)$ m/s^2. If $v(0) = 2$ m/s and $s(0) = 1$ m, find the velocity and position functions $v(t)$ and $s(t)$ of the object.

The mean value theorem is also used to prove inequalities.

EXAMPLE 15: Show that $\ln(1 + x) < x$ for $x > 0$.

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Monotonic Functions and the First Derivative Test

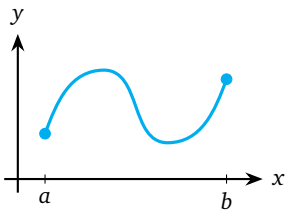
Let us define what it means for a function to be strictly increasing or strictly decreasing on an interval.

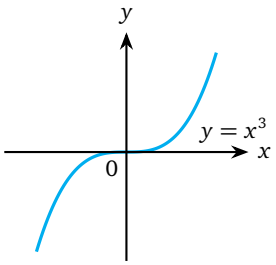
DEFINITION: Let f be defined on an interval I. If $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I, then f is called **strictly increasing** on I; if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in I, it is called **strictly decreasing**. A function that is strictly increasing or strictly decreasing on I is called **strictly monotonic** on I.

NOTE: In some sources, "monotonically increasing" is used for functions with $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$ (non-decreasing); by that definition a constant function is also monotonically increasing. In this course we do not use the words "increasing" and "decreasing" on their own; we always say **strictly** increasing and **strictly** decreasing.

Let us state how the sign of the derivative determines whether a function is strictly increasing or strictly decreasing.

THEOREM: Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) > 0$ for every $x \in (a, b)$, then f is strictly increasing on $[a, b]$; if $f'(x) < 0$ for every $x \in (a, b)$, then f is strictly decreasing on $[a, b]$.





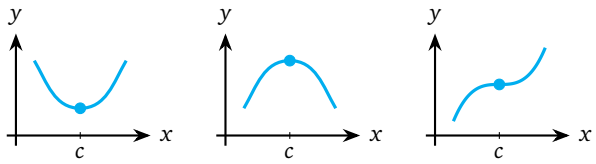
NOTE: If $f'(x) \geq 0$ (or $f'(x) \leq 0$) and f' is zero at only finitely many points, f is still strictly increasing (similarly for strictly decreasing); for example $f(x) = x^3$.

EXAMPLE 16: Find the critical points of $f(x) = 2x^3 - 9x^2 + 12x - 3$ and determine the open intervals on which it is strictly increasing and the open intervals on which it is strictly decreasing.

Let us classify the local extreme values by checking whether the derivative changes sign at a critical point.

THEOREM: Let f be continuous and let c be a critical point of f; also let f be differentiable at every point other than c of an interval containing c. Moving from left to right,

1. if f' changes from negative to positive, then f has a local minimum value at c,
2. if f' changes from positive to negative, then f has a local maximum value at c,
3. if f' does not change sign, then f has no local extreme value at c.



NOTE: At an endpoint of the domain there is only one side; if f' is negative near the endpoint, there is a local minimum at a right endpoint and, if it is positive, a local maximum (the reverse at a left endpoint).

EXAMPLE 17: The function $f(x) = \sqrt{x}(x-3)$ is given on the domain $[0, \infty)$.

a. Find its critical points.

b. Determine the open intervals on which it is strictly increasing and strictly decreasing.

c. Find its local and absolute extreme values.

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EXAMPLE 18: For $f(x) = x^2 e^{-x}$, find the following.

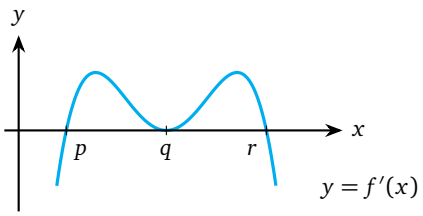
a. The open intervals on which it is strictly increasing and strictly decreasing.

b. Its local and absolute extreme values.

EXAMPLE 19: Find the critical points of $f(x) = x^5 - 5x^3$ and use the first derivative test to determine whether a local extreme value is attained at each of them.

Let us see, through an example, how we read the intervals on which a function is strictly increasing and strictly decreasing from the graph of its derivative.

EXAMPLE 20: The graph of the derivative f' of a differentiable function f is given below. Determine the open intervals on which f is strictly increasing and strictly decreasing and the points of local extrema of f.

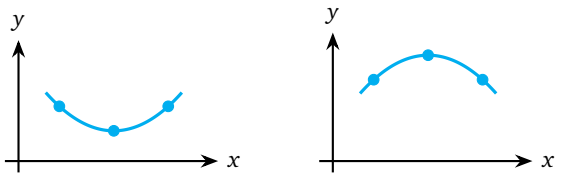


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Concavity and Curve Sketching

Let us examine, with the help of the derivative, which way the graph of a function bends. First, let us define concavity.

DEFINITION: Let f be a differentiable function and let I be an open interval. If f' is strictly increasing on I, the graph of f is called **concave up** on I; if f' is strictly decreasing on I, it is called **concave down**.



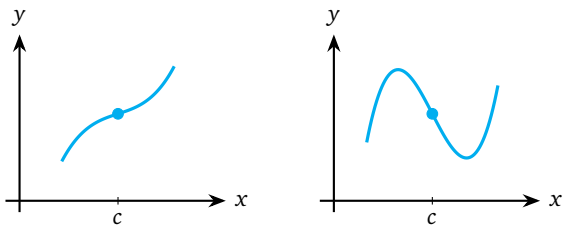
We can also determine concavity from the sign of the second derivative.

THEOREM: Let f be twice differentiable on the open interval I. If $f''(x) > 0$ for every $x \in I$, the graph of f is concave up on I; if $f''(x) < 0$ for every $x \in I$, it is concave down.

EXAMPLE 21: Determine the open intervals on which the graph of $f(x) = x^4 - 6x^2$ is concave up and concave down.

Let us define the points where concavity changes.

DEFINITION: If the graph of f has a tangent at the point $(c, f(c))$ and the concavity changes at this point, then $(c, f(c))$ is called an **inflection point** of the graph.



EXAMPLE 22: Determine whether the graphs of the following functions have an inflection point at the origin $(0, 0)$.

a. $f(x) = x^5$

b. $f(x) = x^6$

c. $f(x) = x^{4/3}$

We can also use the second derivative to classify local extreme values at critical points.

THEOREM: Let f'' be continuous on an open interval containing c.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum value at c,
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum value at c,
3. If $f'(c) = 0$ and $f''(c) = 0$, the test is inconclusive; f may take a local maximum value, a local minimum value, or no extreme value at c.

EXAMPLE 23: Classify the critical points of the following functions using the second derivative test. If the test is inconclusive, use the first derivative test.

a. $f(x) = 2x^3 - 9x^2 + 12x - 3$

b. $f(x) = x^5 - 5x^3$

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Let us combine what we have learned so far and sketch the graph of a function.

NOTE: The graph of $y = f(x)$ is sketched in the following steps:

1. Find the domain and check for symmetry.
2. Find the derivatives f' and f''.
3. Find the critical points and determine the behavior of the function at each.
4. Find the intervals on which the function is strictly increasing and strictly decreasing.
5. Find the inflection points and determine the concavity.
6. Find the asymptotes.
7. Mark the intercepts and the points found above, and sketch the graph.

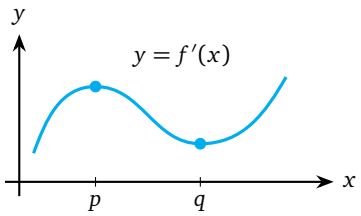
EXAMPLE 24: Sketch the graph of $f(x) = x^4 - 4x^3$; state its extreme values, inflection points and asymptotes.

EXAMPLE 25: Sketch the graph of $f(x) = \frac{x}{x^2+1}$; state its extreme values, inflection points and asymptotes.

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Let us see how to read the concavity of a function from the graph of its derivative.

EXAMPLE 26: The graph of the derivative f' of a differentiable function f is given below. Determine the open intervals on which the graph of f is concave up and concave down, and its inflection points. Assume that the graph continues in both directions as shown.



Indeterminate Forms and L'Hôpital's Rule

Let us examine how to find, using derivatives, the limits of ratios whose numerator and denominator both approach zero or infinity. First, let us name these limits.

DEFINITION: If $f(x)$ and $g(x)$ both approach 0 as $x \to a$, the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

is of the **indeterminate form** $\frac{0}{0}$; if the absolute values of both approach $+\infty$ (each may approach $+\infty$ or $-\infty$), it is of the indeterminate form $\frac{\infty}{\infty}$. In this definition a may also be replaced by a^+, a^-, $+\infty$ or $-\infty$.

Let us state the rule that converts limits of these forms into limits of ratios of derivatives.

THEOREM: L'Hôpital's Rule. Let f and g be differentiable at every point other than a of an open interval containing a, with $g'(x) \neq 0$ at these points. If the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is of the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and the limit

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$

exists (it may be $\pm\infty$), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

The rule also holds for $x \to a^{\pm}$ and $x \to \pm\infty$ (with the interval chosen appropriately).

NOTE: The derivatives of the numerator and the denominator are taken **separately**; the derivative of the quotient $\left(\frac{f}{g}\right)'$ is not used. The rule applies only to the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$; check the form again at every step.

EXAMPLE 27: Evaluate the following limits.

a. $\lim_{x\to 0} \dfrac{e^{3x}-1}{\sin 2x}$

b. $\lim_{x\to 1} \dfrac{x^3-3x+2}{x^4-4x+3}$

c. $\lim_{x\to 0} \dfrac{x-\arctan x}{x^3}$

EXAMPLE 28: Evaluate the following limits.

a. $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt[3]{x}}$

b. $\lim_{x\to\infty} \dfrac{x^3}{e^x}$

c. $\lim_{x\to 0^+} \dfrac{\ln x}{\cot x}$

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Let us convert the other indeterminate forms into ratio form.

NOTE: If $\lim_{x\to a} f(x)\,g(x)$ is of the form $0\cdot\infty$, write $f\cdot g = \frac{f}{1/g}$; if $\lim_{x\to a}\big(f(x)-g(x)\big)$ is of the form $\infty-\infty$, take a common denominator; in either case the limit is brought to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

EXAMPLE 29: Evaluate the following limits.

a. $\lim_{x\to\infty} x\left(e^{1/x}-1\right)$

b. $\lim_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$

We can also bring limits of the forms 1^∞, 0^0 or ∞^0 to ratio form by taking logarithms.

NOTE: For $y = f(x)^{g(x)}$, first find the limit

$$\lim_{x\to a} \ln y = \lim_{x\to a} g(x)\ln f(x) = L;$$

then the limit of y is e^L.

EXAMPLE 30: Evaluate the following limits.

a. $\lim\limits_{x\to 0^+} x^x$

b. $\lim\limits_{x\to 0}(\cos x)^{1/x^2}$

c. $\lim\limits_{x\to\infty}(x^2+1)^{1/\ln x}$

EXAMPLE 31: Find the following limit.

$$\lim_{x\to\infty} \frac{x+\sin x}{x}$$

Does L'Hôpital's Rule give a result for this limit?

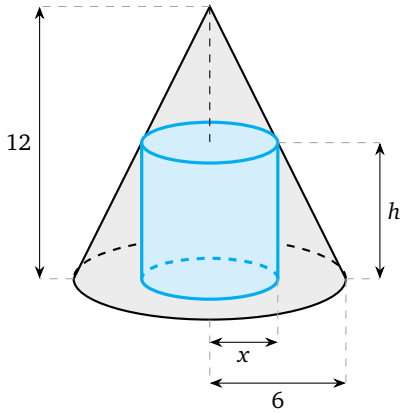
Applied Optimization

Let us solve, using derivatives, problems that require finding the greatest or least value of a quantity. An applied optimization problem is solved in the following steps:

1. Read the problem; identify the given quantities and the quantity to be maximized or minimized, and draw a figure.
2. Name the variables and write the relations among them.
3. Write the quantity as a function of a single variable and determine its domain.
4. Examine the critical points and the endpoints; state the result as the answer to the question. If the domain is not a closed interval and f' changes sign at the only critical point, or f'' keeps the same sign on the whole interval, that point is an absolute extremum; if the domain includes an endpoint, its value is also compared.

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EXAMPLE 32: A right circular cylinder is placed inside a right circular cone of base radius 6 and height 12, with the base of the cylinder on the base of the cone. The cone and the cylinder are shown in the figure; the radius of the cylinder is denoted by x and its height by h. Find the radius, height and volume of the cylinder of greatest volume.



Let us derive a result used in optimization problems in economics.

NOTE: Let $r(x)$ be the revenue, $c(x)$ the cost and $P(x) = r(x) - c(x)$ the profit. If the profit is greatest at an interior point, then $P'(x) = 0$, that is, $r'(x) = c'(x)$: marginal revenue equals marginal cost.

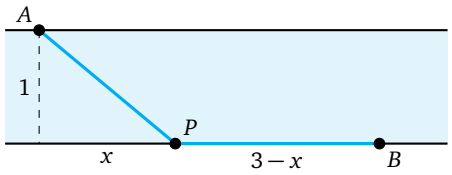
EXAMPLE 33: When x units of a product are sold, the price per unit is $p(x) = 120 - 2x$ liras. The cost of production is $c(x) = 300 + 20x$ liras.

a. Find the production level that maximizes the profit, the unit price and the profit.

b. Answer the same question if at most 20 units can be produced.

EXAMPLE 34: The cost of producing x hundred units of a product is $c(x) = 2x^2 + 18x + 98$ thousand liras. Find the production level that minimizes the average (unit) cost; compare the average cost with the marginal cost at this level.

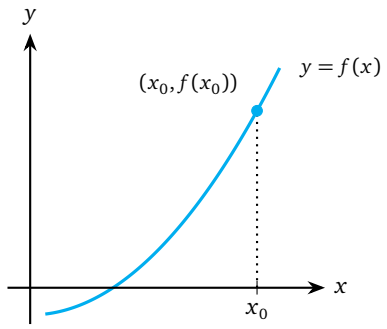
EXAMPLE 35: A straight river is 1 km wide. On one bank there is a point A; on the opposite bank there is a point B, 3 km downstream from the point directly across from A. A pipeline will be laid from A to B, running in the river from A to a point P on the opposite bank and then along the bank to B. The cost of the pipeline per kilometer is 5 thousand liras in the river and 3 thousand liras on land. How many km from the point opposite A should P be, and what is the minimum cost?



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Newton's Method

We can often find the roots of equations whose solutions cannot be given by a formula to any desired accuracy by using tangent lines. First, let us derive the formula of the method on a graph.



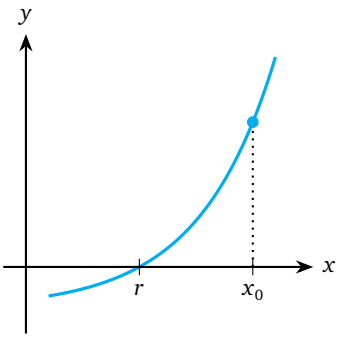
A root of the equation $f(x) = 0$ is found approximately by **Newton's method** in the following steps:

1. Choose an x_0 close to the root (a graph or the intermediate value theorem helps).
2. With $f'(x_n) \neq 0$, compute $x_1, x_2, \ldots$ using
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
3. Stop when x_{n+1} and x_n agree to the desired number of digits.

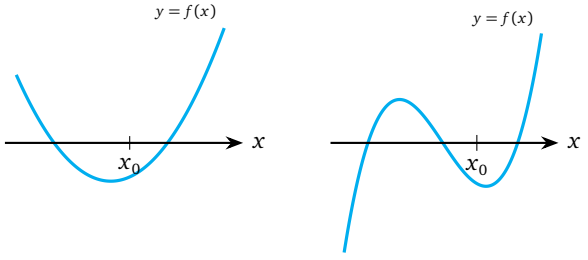
EXAMPLE 36: Apply Newton's method to the equation $x^3 - 10 = 0$ starting with $x_0 = 2$ to approximate $\sqrt[3]{10}$. Compute x_1 and x_2 to four decimal places.

EXAMPLE 37: For $a > 0$, apply Newton's method to the equation $x^2 - a = 0$ (with $f(x) = x^2 - a$) and write a simplified formula for x_{n+1}. Use this formula to find x_2 for $a = 7$ and $x_0 = 3$.

EXAMPLE 38: Below is the graph of a strictly increasing, concave up function f, with root r and initial value $x_0 > r$. Show the values x_1 and x_2 of Newton's method on the graph. On which side of r do $x_1, x_2, \ldots$ lie, and how does the sequence behave?



Newton's method does not always converge to the root we are looking for: the starting point and the shape of the curve determine which root the sequence approaches. Let us start the method with x_0 in the graphs below.



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Antiderivatives

Let us consider the process of finding a function whose derivative is known. First, let us name the function we find.

DEFINITION: Let I be an interval. If $F'(x) = f(x)$ for every $x \in I$, then F is called an **antiderivative** of f on I.

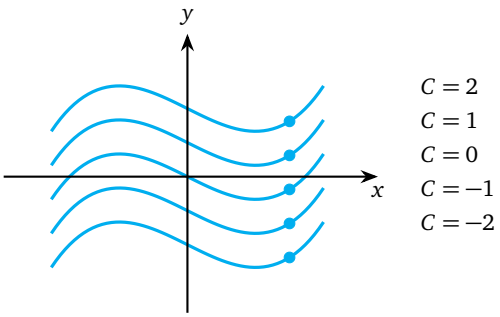
EXAMPLE 39: Find an antiderivative of each of the following functions.

a. $f(x) = 4x^3$

b. $g(x) = \sec^2 x$

c. $h(x) = \dfrac{1}{1+x^2} + 3e^{3x}$

The graphs of several antiderivatives of $f(x) = x^2 - 1$ are shown below; let us examine how the antiderivatives of a function are related to one another.



THEOREM: If F is an antiderivative of f on an interval I, then every antiderivative of f on I has the form $F(x) + C$ for some constant C.

Let us define the notation that lets us write all the antiderivatives of a function in a single expression.

DEFINITION: The set of all antiderivatives of f on I is called the **indefinite integral** of f and is denoted by $\int f(x)\,dx$. If F is an antiderivative of f, we write

$$\int f(x)\,dx = F(x) + C$$

Here $f(x)$ is the **integrand**, x is the **variable of integration** and C is the **constant of integration**.

NOTE: Differentiation and the indefinite integral work as inverses of each other:

$$\frac{d}{dx}\int f(x)\,dx = f(x), \qquad \int F'(x)\,dx = F(x) + C.$$

Let us write the basic integrals by reading the familiar derivative formulas backwards. In the table, $k \neq 0$ is a constant.

Integral	Result
$\int x^n\,dx$	$\dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
$\int \sin kx\,dx$	$-\dfrac{1}{k}\cos kx + C$
$\int \cos kx\,dx$	$\dfrac{1}{k}\sin kx + C$
$\int \sec^2 kx\,dx$	$\dfrac{1}{k}\tan kx + C$
$\int \csc^2 kx\,dx$	$-\dfrac{1}{k}\cot kx + C$
$\int \sec kx \tan kx\,dx$	$\dfrac{1}{k}\sec kx + C$
$\int \csc kx \cot kx\,dx$	$-\dfrac{1}{k}\csc kx + C$
$\int e^{kx}\,dx$	$\dfrac{1}{k}e^{kx} + C$
$\int a^{kx}\,dx$	$\dfrac{a^{kx}}{k \ln a} + C \quad (a > 0,\ a \neq 1)$
$\int \dfrac{dx}{x}$	$\ln\lvert x\rvert + C$ $(x \neq 0,$ on each interval$)$
$\int \dfrac{dx}{\sqrt{1-k^2x^2}}$	$\dfrac{1}{k}\arcsin kx + C \quad (\lvert kx\rvert < 1)$
$\int \dfrac{dx}{1+k^2x^2}$	$\dfrac{1}{k}\arctan kx + C$
$\int \dfrac{dx}{x\sqrt{x^2-1}}$	$\operatorname{arcsec}\lvert x\rvert + C \quad (\lvert x\rvert > 1)$

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Let us state how the indefinite integral behaves with respect to sums and constant multiples.

NOTE: For a constant k,

$$\int kf(x)\,dx = k\int f(x)\,dx,$$

$$\int \big(f(x)\pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

hold. The constants of the separate integrals are combined into a single constant C.

EXAMPLE 40: Evaluate the following indefinite integrals.

a. $\int x^{-5}\,dx$

b. $\int \sqrt[4]{x}\,dx$

c. $\int \cos 4x\,dx$

d. $\int \sec^2\frac{x}{3}\,dx$

e. $\int \frac{dx}{\sqrt{1-9x^2}}$

f. $\int 10^{2x}\,dx$

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EXAMPLE 41: Evaluate the following indefinite integrals by first rewriting the integrand in a suitable form.

a. $\int \frac{x^3-2\sqrt{x}}{x^2}\,dx \quad (x>0)$

b. $\int \cos^2 x\,dx$

c. $\int \tan^2 3x\,dx$

EXAMPLE 42: Show that the following equalities are true by differentiating.

a. $\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1}+C$

b. $\int \ln x\,dx = x\ln x - x + C \quad (x>0)$

Initial Value Problems

Let us define the problems of finding a function whose derivative and value at a point are given.

DEFINITION: An equation containing a function and its derivatives is called a **differential equation**; in this section we consider equations of the types $\frac{dy}{dx} = f(x)$ and $\frac{d^2y}{dx^2} = f(x)$. The solutions of a first-order equation are the antiderivatives of f: $y = F(x)+C$ (the **general solution**); the general solution of a second-order equation contains two constants. If an **initial condition** $y(x_0) = y_0$ is given together with the equation, the problem is called an **initial value problem** and a solution satisfying the condition is called a **particular solution**. Problems involving the second derivative are given two initial conditions.

EXAMPLE 43: Solve the following initial value problem on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

$$\frac{dy}{dx} = \sec^2 x + 3x^2, \quad y(0) = 2$$

EXAMPLE 44: Solve the following initial value problem.

$$\frac{d^2y}{dx^2} = \sin x + 6x, \quad y'(0) = 2,\ y(0) = 1$$

Antiderivatives in rectilinear motion

Let us find, from the acceleration of an object moving along a straight line, first its velocity and then its position.

EXAMPLE 45: A stone is thrown vertically upward from the ground with an initial speed of 25 m/s. Its acceleration is constant, 10 m/s^2 directed downward (there is no air resistance). Taking the upward direction as positive, find the following.

a. The maximum height the stone reaches.

b. The time the stone hits the ground and its velocity at that moment.

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Exercises and Problems

1. The absolute maximum value of $f(x) = x^3 - 12x + k$ on the interval $[-3, 3]$ is 15. Find the constant k and the absolute minimum value of f on this interval.

2. Let f be differentiable for every x, with $f(1) = -2$ and $1 \leq f'(x) \leq 4$ for every x. Using the mean value theorem, find a lower and an upper bound for $f(5)$.

3. For which values of k is $f(x) = x^3 + kx^2 + 3x + 1$ strictly increasing on $\mathbb{R}$?

4. The graph of $f(x) = x^3 + ax^2 + bx + c$ is known to have an inflection point at $(1, -1)$, and f is known to have a local extreme value at $x = 2$. Find the constants a, b and c.

5. For what value of a is the following limit a finite real number? Evaluate the limit for this value of a.

$$\lim_{x \to 0} \frac{ax - \sin x}{x^3}$$

6. A wire of length L is cut into two pieces; a square is formed from one piece and a circle from the other. How should the wire be cut so that the sum of the areas of the square and the circle is least, and so that it is greatest? Find the least and greatest total areas in terms of L.

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7. Sketch the graph of the following function by examining its asymptotes, the intervals on which it is strictly increasing and strictly decreasing, its local extreme values and its concavity.

$$f(x)=\frac{x^2}{x-1}$$

Answers

Example questions

1	a. min. 0, no max. b. max. 9, min. 0 c. max. 9, no min. d. neither
2	abs. max. $x=r$, abs. min. $x=q$ local max. $x=p,r$, local min. $x=a,q,b$
3	a. $x=0,3$ b. $x=0,2$ c. $x=-1,0,1$
4	max. 3 $(x=1,4)$ min. -1 $(x=0,3)$
5	max. $4-4\ln 2$ $(x=4)$ min. $2-2\ln 2$ $(x=2)$
6	max. $\sqrt{3}+\frac{\pi}{6}$ $(x=\frac{\pi}{6})$ min. $\frac{5\pi}{6}-\sqrt{3}$ $(x=\frac{5\pi}{6})$
7	abs. max. 5 $(x=\pm 3)$, abs. min. 0 $(x=\pm 2)$ local max. 5 $(x=\pm 3)$, 4 $(x=0)$ local min. 0 $(x=\pm 2)$
8	$f(x)=2x-\cos x$: $f(0)=-1$, $f\left(\frac{\pi}{2}\right)=\pi$ $f'(x)=2+\sin x>0$
9	$c=\pm\frac{2}{\sqrt{3}}$
10	$c=\sqrt{\frac{13}{3}}$
11	$\frac{45-5}{20}=2$ cm/day
12	$f(x)=\frac{\pi}{2}$
13	$f(x)=\ln x+x^2+2$
14	$v(t)=3t^2-4t+2$ $s(t)=t^3-2t^2+2t+1$
15	Go to the video.
16	critical: $x=1,2$ strictly decreasing: $(1,2)$ strictly increasing: $(-\infty,1)$, $(2,\infty)$
17	critical: $x=1$ local max. $f(0)=0$ strictly decreasing: $(0,1)$ strictly increasing: $(1,\infty)$ local and abs. min. $f(1)=-2$, no abs. max.

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18 strictly decreasing: $(-\infty, 0)$, $(2, \infty)$
strictly increasing: $(0, 2)$
local and abs. min. $f(0) = 0$
local max. $f(2) = 4e^{-2}$, no abs. max.

19 critical: $x = -\sqrt{3}, 0, \sqrt{3}$ no extremum at $x = 0$
local max. $f(-\sqrt{3}) = 6\sqrt{3}$
local min. $f(\sqrt{3}) = -6\sqrt{3}$

20 strictly decreasing: $(-\infty, p)$, (r, ∞)
strictly increasing: (p, r)
local min. $x = p$, local max. $x = r$
no extremum at $x = q$

21 concave up: $(-\infty, -1)$, $(1, \infty)$
concave down: $(-1, 1)$

22 a. yes b. no c. no

23 a. local max. $f(1) = 2$, local min. $f(2) = 1$
b. local max. $f(-\sqrt{3}) = 6\sqrt{3}$
local min. $f(\sqrt{3}) = -6\sqrt{3}$
test inconclusive at $x = 0$, no extremum

24 abs. min. $f(3) = -27$, no abs. max.
inflection points: $(0, 0)$, $(2, -16)$ no asymptotes

25 abs. max. $f(1) = \frac{1}{2}$, abs. min. $f(-1) = -\frac{1}{2}$
inflection points: $x = 0, \pm\sqrt{3}$ asymptote: $y = 0$

26 concave up: $(-\infty, p)$, (q, ∞)
concave down: (p, q) inflection points: $x = p, q$

27 a. $\frac{3}{2}$ b. $\frac{1}{2}$ c. $\frac{1}{3}$

28 a. 0 b. 0 c. 0

29 a. 1 b. $\frac{1}{2}$

30 a. 1 b. $e^{-1/2}$ c. e^2

31 limit: 1 the rule gives no result:
$\frac{(x+\sin x)'}{x'} = 1 + \cos x$ has no limit

32 $x = 4$, $h = 4$ $V = 64\pi$

33 a. $x = 25$, $p = 70$ liras, profit 950 liras
b. $x = 20$, $p = 80$ liras, profit 900 liras

34 $x = 7$ (hundred units)
avg. cost $= c'(7) = 46$ (thousand liras/hundred)

35 $x = \frac{3}{4}$ km minimum cost: 13 thousand liras

36 $x_1 = \frac{13}{6} \approx 2.1667$ $x_2 \approx 2.1545$

37 $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$
$x_1 = \frac{8}{3}$, $x_2 = \frac{127}{48} \approx 2.6458$

38 $x_1, x_2, \ldots > r$; the sequence decreases to r

39 a. x^4 b. $\tan x$ c. $\arctan x + e^{3x}$

40 a. $-\frac{1}{4x^4} + C$ b. $\frac{4}{5}x^{5/4} + C$ c. $\frac{1}{4}\sin 4x + C$
d. $3\tan\frac{x}{3} + C$ e. $\frac{1}{3}\arcsin 3x + C$ f. $\frac{10^{2x}}{2\ln 10} + C$

41 a. $\frac{x^2}{2} + \frac{4}{\sqrt{x}} + C$ b. $\frac{x}{2} + \frac{\sin 2x}{4} + C$
c. $\frac{1}{3}\tan 3x - x + C$

42 a. $\left(\sqrt{x^2+1}\right)' = \frac{x}{\sqrt{x^2+1}}$ b. $(x\ln x - x)' = \ln x$

43 $y = \tan x + x^3 + 2$

44 $y = x^3 + 3x + 1 - \sin x$

45 a. 31.25 m b. $t = 5$ s, $v = -25$ m/s

Exercises and problems

1 $k = -1$ absolute minimum: -17

2 $2 \le f(5) \le 14$

3 $-3 \le k \le 3$

4 $a = -3$, $b = 0$, $c = 1$

5 $a = 1$ limit: $\frac{1}{6}$

6 least: square piece $\frac{4L}{\pi+4}$, area $\frac{L^2}{4(\pi+4)}$
greatest: whole wire for the circle, area $\frac{L^2}{4\pi}$

7 asymptotes: $x = 1$ (vertical), $y = x + 1$ (oblique)
local max. $f(0) = 0$, local min. $f(2) = 4$

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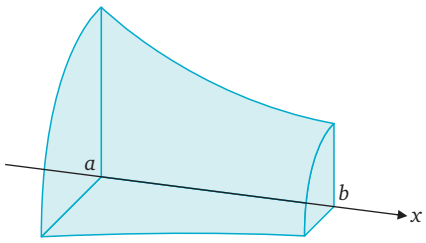
CHAPTER 5

APPLICATIONS OF INTEGRATION

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Volume Calculation by Cross-Sectional Area

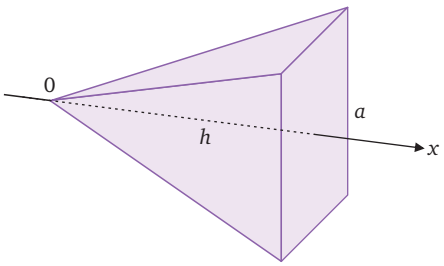
Let's discuss how to calculate volume by slicing with parallel surfaces. Let's define the mathematical definition of volume.



DEFINITION: The volume of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b:

$$V = \int_a^b A(x)\,dx$$

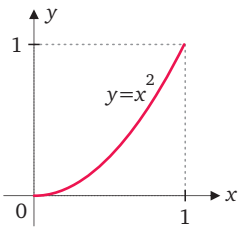
EXAMPLE 1: Calculate the volume of a square right pyramid with base edge length a and height h.



Solids of revolution: disk method

Let's examine what solids of revolution are and how they are obtained. Let's explain the disk method used in volume calculation and express it mathematically.

EXAMPLE 2: The region between the curve $y = x^2$, the interval $0 \le x \le 1$, and the x-axis is rotated around the x-axis to form a solid. Calculate the volume of this solid of revolution using an integral.



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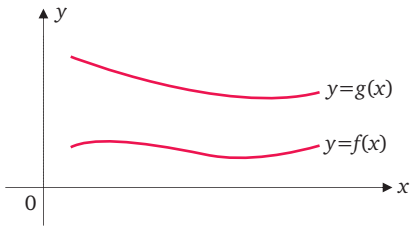
EXAMPLE 3: Consider the region bounded by the curve $y = \sqrt{x}$, the line $y = 2$, and the line $x = 0$. Write the integral that gives the volume of the solid of revolution formed by rotating this region around the line $y = 2$.

Let's mathematically express the volume calculation for the case of rotation around the y-axis.

EXAMPLE 4: Consider the region between the parabola $x = y^2$ and the line $x = 1$. Write the integral that gives the volume of the solid of revolution formed by rotating this region around the line $x = 1$.

Solids of revolution: washer method

Let us examine how to calculate the volumes of solids of revolution with hollow centers using the washer method.



EXAMPLE 5: Consider the region bounded by the curve $y = e^x$, the line $y = x + 1$, and the line $x = 1$. Write the integral that gives the volume of the solid of revolution formed by rotating this region around the x-axis.

EXAMPLE 6: Consider the region remaining between the parabola $x = y^2$ and the line $x = 1$. Write the integral that gives the volume of the solid of revolution formed by rotating this region around the line $x = 2$.

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Volume Calculation with Cylindrical Shells

Let's obtain a solid of revolution by rotating the region between the parabola $y = 4 - x^2$ and the x-axis around the line $x = 3$. Let's discuss how we can find the volume of this solid.

Let's express the cylindrical shell method mathematically.

EXAMPLE 7: Consider the region between the curve $y = \sqrt{x}$, the x-axis, and the line $x = 1$. Write the integral that gives the volume of the solid of revolution formed by rotating this region around the y-axis using the cylindrical shell method.

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EXAMPLE 8: Write the integral that gives the volume of the solid formed by rotating the region between the curve $x = y - y^2$ and the y-axis around the x-axis using the cylindrical shell method.

Arc Length

Let's discuss how the length of a curve is obtained. Then, let's define the length of a curve.

DEFINITION: If the derivative f' is continuous on the closed interval $[a, b]$, the length of the curve $y = f(x)$ from point $A = (a, f(a))$ to point $B = (b, f(b))$ is the value of the following integral:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

EXAMPLE 9: Write the integral that gives the length of the curve $y = x^2 - x$ in the interval from $x = 0$ to $x = 2$.

EXAMPLE 10: Find the length of the curve $y = \frac{2}{3}x^{3/2}$ in the interval from $x = 0$ to $x = 3$.

Areas of Surfaces of Revolution

Let's discuss how we obtain the surface area of solids of revolution. Then, let's examine the mathematical definition of surface area for solids of revolution.

DEFINITION: If the function $f(x) \geq 0$ is continuously differentiable on the interval $[a, b]$, the surface area formed by rotating the graph of $y = f(x)$ around the x-axis is the value of the following integral:

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

EXAMPLE 11: Find the area of the surface formed by rotating the part of the curve $y = \sqrt{x}$ in the interval $1 \leq x \leq 2$ around the x-axis.

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Physical Work and Pressure Force Calculations

Let's define the total work done by a variable force $F(x)$ while moving an object from point $x = a$ to point $x = b$.

EXAMPLE 12: Find the work required to compress a spring with a natural length of 40 cm and a force constant of $k = 300$ N/m to a length of 20 cm.

EXAMPLE 13: A rope 10 m long with a mass per unit length of 0.1 kg/m is hanging from the top of a building. How many joules of work are required to pull this entire rope to the top of the building? (Take the magnitude of the gravitational acceleration as $g = 10$ m/s^2.)

EXAMPLE 14: The water in a full rectangular prism-shaped pool with dimensions 8 m × 12 m and a depth of 2 m will be emptied by pumping it to the top edge level of the pool. Assuming the specific weight of water is 10 kN/m^3, how many joules of work are required for this process?

Let's examine how we can utilize integrals when calculating pressure force.

EXAMPLE 15: A flat isosceles triangular plate with a base of 3 m and a height of 2 m is placed vertically in water with its base at the top. The upper edge of the plate (the triangle base) is 1 m below the water surface. Assuming the specific weight of water is 10 kN/m^3, find the total force exerted by the water on one side of the plate.

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Moments and Center of Mass

Let's discuss how to find the center of mass of a one-dimensional object and the concept of moments. Let's express the position of the center of mass mathematically.

EXAMPLE 16: For a rod located on the x-axis in the interval $[0, 2]$ with a position-dependent mass density $\delta(x) = 4 + 2x^3$;

a. Find its total mass.

b. Find the coordinate of its center of mass.

Let's discuss how to find the center of mass of a two-dimensional object. Let's express the position of the center of mass mathematically.

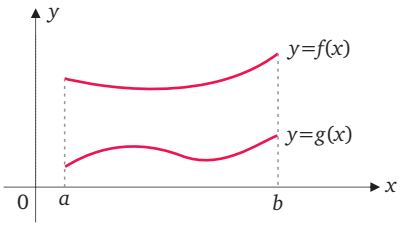
EXAMPLE 17: There is a triangular plate in the coordinate plane bounded by the x-axis, the line $x = 2$, and the line $y = 3x$. Given that the position-dependent density of this plate is $\delta(x) = x + 2$;

a. Find the total mass of the plate.

b. Find the x-coordinate of the center of mass of the plate.

c. Find the y-coordinate of the center of mass of the plate.

Let's discuss how to find the center of mass of plates remaining between two curves.



EXAMPLE 18: In the coordinate plane, there is a plate bounded from above by the curve $f(x) = \sqrt{x}$ and from below by the curve $g(x) = x^2$. Given that the position-dependent density function of this plate is $\delta(x) = 6x$;

a. Find the total mass of the plate.

b. Find the x-coordinate of the center of mass of the plate.

c. Find the y-coordinate of the center of mass of the plate.

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Centroids

Let us discuss how to find the centroid of a shape.

EXAMPLE 19: There is a plate in the coordinate plane located above the x-axis, bounded by a semicircle of radius $r = 4$ centered at the origin and the x-axis. The density of this plate is equal to the constant δ. For this plate;

a. Find its total mass.

b. Find the coordinates of its center.

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Exercises and Problems

1. In the first quadrant, consider the region bounded on the left by the circle $x^2 + y^2 = 4$, on the right by the line $x = 2$, and from above by the line $y = 2$. Write the integral that gives the volume of the solid of revolution formed by rotating this region about the y-axis;

a. Using the washer method,

b. Using the cylindrical shells method.

2. Consider the region bounded by the parabola $y = x^2 + 1$ and the line $y = -x + 3$. Write the integral that gives the volume of the solid of revolution formed by rotating this region about the x-axis.

3. To calculate the volume of the solid of revolution formed by rotating the triangular region with vertices $(1,0)$, $(2,1)$, and $(1,1)$ about the y-axis;

a. Write the integral giving the volume using the disk/washer method.

b. Write the integral giving the volume using the cylindrical shells method.

4. Find the length of the graph of the function $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ over the interval $1 \leq x \leq 3$.

5. Find the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ over the interval from $x = 0$ to $x = \ln(3)$.

6. Write the integral required to find the circumference of a circle with radius r centered at the origin.

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7. Let S be the surface obtained by rotating the part of the curve $y = \frac{1}{3}x^3$ on the interval $0 \leq x \leq 1$ about the x-axis. Calculate the area of this surface.

8. Let region D be a region bounded by the curves $y = e^x$, $y = \sqrt{x+1}$ and the line $y = 2$.

a. Express the volume of the solid S formed by rotating region D about the y-axis using one or more integrals with the slicing (disk/washer) method.

b. Express the volume of the solid S given in part (a) using one or more integrals with the cylindrical shells method.

c. Express the volume of the solid formed by rotating region D about the line $x = -2$ using one or more integrals with the slicing (disk/washer) method.

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9. The region remaining inside the circle given by the equation $x^2-6x+y^2=-5$ is rotated about the y-axis. Write the integral that gives the volume of the solid formed as a result of this rotation using the cylindrical shells method.

10. Let region $\mathscr{R}$ be the region bounded by the lines $x=0$ and $y=x-3$ and the curve $y=(x-3)^2(x+1)$.

a. Express the volume of the solid formed by rotating region $\mathscr{R}$ about the y-axis as a definite integral.

b. Express the volume of the solid formed by rotating region $\mathscr{R}$ about the line $y=-3$ as a definite integral.

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Answers

Example questions

1 $V=\frac{1}{3}a^2h$

2 $V=\frac{\pi}{5}$

3 $V=\pi\int_0^4(2-\sqrt{x})^2dx$

4 $V=\pi\int_{-1}^1(1-y^2)^2\,dy$

5 $V=\pi\int_0^1\left[e^{2x}-(x+1)^2\right]dx$

6 $V=\pi\int_{-1}^1\left[(2-y^2)^2-1\right]dy$

7 $V=\int_0^1 2\pi x\sqrt{x}\,dx$

8 $V=2\pi\int_0^1(y^2-y^3)\,dy$

9 $L=\int_0^2\sqrt{1+(2x-1)^2}\,dx$

10 $L=\frac{14}{3}$

11 $S=\frac{\pi}{6}(27-5\sqrt{5})$

12 $W=6\text{ J}$

13 $W=50\text{ J}$

14 $W=1920\text{ kJ}$

15 $F=50\text{ kN}$

16 a. $M=16$ b. $\bar{x}=1.3$

17 a. $M=20$ b. $\bar{x}=1.4$ c. $\bar{y}=2.1$

18 a. $M=0.9$ b. $\bar{x}=\frac{4}{7}$ c. $\bar{y}=\frac{5}{9}$

19 a. $M=8\pi\delta$ b. $(0,\frac{16}{3\pi})$

Exercises and problems

1

a. $V=\pi\int_0^2 y^2\,dy$

b. $V=2\pi\int_0^2 x\left(2-\sqrt{4-x^2}\right)dx$

2 $V=\pi\int_{-2}^1\left[(-x+3)^2-(x^2+1)^2\right]dx$

3

a. $V=\pi\int_0^1\left[(y+1)^2-1\right]dy$

b. $V=2\pi\int_1^2 x(2-x)dx$

4 $L = \frac{14}{3}$

5 $L = \frac{4}{3}$

6 $C = 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx$

7 $A = \frac{\pi}{9}(2\sqrt{2}-1)$

8
a. $V = \pi\int_1^2 \left[(y^2-1)^2-(\ln y)^2\right] dy$

b. $V = 2\pi\int_0^{\ln 2} x\left(e^x-\sqrt{x+1}\right) dx$
$+2\pi\int_{\ln 2}^3 x\left(2-\sqrt{x+1}\right) dx$

c. $V = \pi\int_1^2 \left[(y^2+1)^2-(\ln y+2)^2\right] dy$

9 $V = 4\pi\int_1^5 x\sqrt{4-(x-3)^2}\,dx$

10
a. $V = 2\pi\int_0^3 x\left[(x-3)^2(x+1)-(x-3)\right] dx$

b. $V = \pi\int_0^3 \left(\left[(x-3)^2(x+1)+3\right]^2-x^2\right) dx$

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CHAPTER 7

TECHNIQUES OF INTEGRATION

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Using Basic Integration Formulas

Let's examine the basic integration formulas.

1. $\int k\,dx = kx + C$ (for any k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

EXAMPLE 1: Evaluate the indefinite integrals given below using appropriate substitution methods and basic integral formulas:

a. $\int \frac{e^x}{9 + e^{2x}}\,dx$

b. $\int \frac{1}{\sqrt{4 - 25x^2}}\,dx$

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EXAMPLE 2: Evaluate the following integral using the method of completing the square:

$$\int \frac{dx}{\sqrt{3 - 2x - x^2}}$$

EXAMPLE 3: Evaluate the following indefinite integral:

$$\int \frac{1}{1+\cos x}\,dx$$

EXAMPLE 4: Evaluate the following indefinite integral:

$$\int \frac{4x^2-5x+3}{2x-1}\,dx$$

EXAMPLE 5: Evaluate the following indefinite integral:

$$\int \frac{dx}{(2+\sqrt{x})^2}$$

Integration by Parts

Let's derive the integration by parts formula. Let's write the differential form of the formula.

EXAMPLE 6: Evaluate the following indefinite integral:

$$\int x \sin x\,dx$$

EXAMPLE 7: Evaluate the following indefinite integral:

$$\int \ln x\,dx$$

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EXAMPLE 8: Evaluate the following indefinite integral:

$$\int x^2 e^x \, dx$$

EXAMPLE 9: Evaluate the following indefinite integral:

$$\int e^x \sin x \, dx$$

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EXAMPLE 10: Derive a formula that expresses the following integral in terms of an integral of a lower power of $\cos x$:

$$\int \cos^n x \, dx$$

Let's examine how to evaluate definite integrals using the integration by parts method.

EXAMPLE 11: Consider the region bounded by the curve $y = xe^{-x}$, the x-axis, and the vertical line $x = 5$;

a. Express its area as a definite integral.

b. Evaluate the integral using integration by parts.

Trigonometric Integrals

Let m and n be integers; let's examine the solution strategies for the following integral:

$$\int \sin^m x \cos^n x \, dx$$

EXAMPLE 12: Evaluate the following indefinite integral:

$$\int \cos^3 x \sin^2 x \, dx$$

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EXAMPLE 13: Evaluate the following indefinite integral:

$$\int \cos^3 x \, dx$$

EXAMPLE 14: Evaluate the following indefinite integral:

$$\int \sin^4 x \cos^2 x \, dx$$

Let's discuss how to evaluate integrals involving powers of $\tan x$ and $\sec x$ functions.

EXAMPLE 16: Evaluate the following indefinite integral.

$$\int \tan^5 x \, dx$$

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Let's examine how to eliminate the radical in trigonometric integrals containing radical expressions.

EXAMPLE 15: Evaluate the following indefinite integral.

$$\int \sqrt{1-\cos x}\, dx$$

EXAMPLE 17: Evaluate the following indefinite integral.

$$\int \sec^3 x \, dx$$

We can use product-to-sum formulas when evaluating the integrals $\int \sin mx \sin nx \, dx$, $\int \sin mx \cos nx \, dx$, and $\int \cos mx \cos nx \, dx$.

- $\sin x \cdot \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$
- $\cos x \cdot \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$
- $\sin x \cdot \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

EXAMPLE 18: Evaluate the following indefinite integral:

$$\int \sin x \cos 3x \, dx$$

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Trigonometric Substitutions

Let's examine how to apply trigonometric substitutions through examples.

EXAMPLE 19: Evaluate the following indefinite integral:

$$\int \frac{1}{1+x^2} dx$$

EXAMPLE 20: Evaluate the following indefinite integral for $a > 0$:

$$\int \frac{dx}{\sqrt{a^2+x^2}}$$

EXAMPLE 21: Evaluate the following indefinite integral:

$$\int \frac{\sqrt{16-x^2}}{x^2}\,dx$$

Integration of Rational Functions by Partial Fractions

Let's examine how to apply the method of partial fraction decomposition through examples.

EXAMPLE 22: Evaluate the following indefinite integral:

$$\int \frac{7x+1}{(x-2)(x+3)}dx$$

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EXAMPLE 23: Evaluate the following indefinite integral:

$$\int \frac{3x-5}{(x^2+1)(x+1)^2}dx$$

Improper Integrals

Let's discuss how integrals with infinite limits can yield finite results through an example.

EXAMPLE 24: Consider the region under the curve $y = \frac{1}{x^2}$ extending from the line $x = 1$ to infinity to the right ($x \geq 1$). To determine whether this region has a finite area;

a. Find the function $A(b)$ that gives the area of the part of the region between $x = 1$ and $x = b$ ($b > 1$).

b. Find the total area by calculating the limit of this area as $b \to \infty$.

Let's define **improper integrals of Type 1**. Let's discuss what it means for these integrals to be **convergent** or **divergent**.

EXAMPLE 25: Is the area under the function $f(x) = xe^{-x}$ on the interval from $x = 1$ to $x = \infty$ finite? If so, find the value of this area.

$$\int_1^{\infty} \frac{dx}{x^p}$$

Determine for which values of p the integral is convergent or divergent.

Let's discuss how the presence of a vertical asymptote in the integrand can lead to improper integrals. Let's examine the following integral:

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

Let's define **improper integrals of Type 2**. Let's discuss what it means for these integrals to be convergent or divergent.

EXAMPLE 26: Determine whether the improper integral below converges or diverges:

$$\int_0^2 \frac{1}{2-x}\, dx$$

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EXAMPLE 27: Determine whether the following improper integral converges or diverges:

$$I = \int_0^2 \frac{dx}{\sqrt{|x-1|}}$$

EXAMPLE 28: Use the Direct Comparison Test to determine whether each improper integral converges or diverges.

a. $\int_1^\infty \frac{1}{x^3+1}\,dx$

b. $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

c. $\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx$

d. $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$

Convergence and divergence tests

Let's explain the **Direct Comparison Test**.

THEOREM: Let functions f and g be continuous on the interval $[a,\infty)$ and $0 \le f(x) \le g(x)$ for all $x \ge a$. In this case:

a. If $\int_a^\infty g(x)\,dx$ is convergent, then $\int_a^\infty f(x)\,dx$ is also convergent.

b. If $\int_a^\infty f(x)\,dx$ is divergent, then $\int_a^\infty g(x)\,dx$ is also divergent.

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Let's explain the **Limit Comparison Test**.

THEOREM: Let f and g be positive continuous functions on the interval $[a,\infty)$. If,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

is satisfied;

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both integrals either converge or both diverge.

EXAMPLE 29: Use the Limit Comparison Test to determine whether each improper integral converges or diverges.

a. $\int_2^\infty \frac{1}{x^2-1}\,dx$ b. $\int_1^\infty \frac{x+2}{2x^2+1}\,dx$

Exercises and Problems

1. Evaluate the following indefinite integrals.

a. $\int \frac{dx}{\sqrt{1-4x^2}}$ c. $\int \frac{e^{-\tan x}}{\cos^2 x}dx$

b. $\int (\sec x + \tan x)^2 dx$ d. $\int \frac{dx}{e^{2x}+e^{-2x}}$

2. Evaluate the following integrals using the method of integration by parts.

a. $\int x^2 \ln(x)\,dx$ c. $\int e^{3x}\cos x\,dx$

b. $\int (2x-3)\sin 2x\,dx$ d. $\int x^2 e^{-x}\,dx$

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3. Let n be a positive integer; show that the following equality holds using the method of integration by parts:

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

4. Let f be a continuous function; show that the following equality holds using the method of integration by parts:

$$\int_a^b \left(\int_x^b f(t)\,dt\right)dx = \int_a^b (x-a)f(x)\,dx$$

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5. Evaluate the following trigonometric integrals.

a. $\int \sin^5 x\,dx$

b. $\int \frac{\sin^2 x}{\sqrt{1-\cos x}}dx$

c. $\int \tan^4 x \sec^4 x\,dx$

d. $\int x \sin^3 x\,dx$

6. Evaluate the following integrals using trigonometric substitutions.

a. $\int \frac{dx}{x\sqrt{x^2-1}}$

b. $\int \sqrt{\frac{x+1}{1-x}}\,dx$

c. $\int \frac{dx}{(9-x^2)^{3/2}}$

d. $\int \frac{2dx}{(4x^2+1)^2}$

7. Evaluate the following integrals using the method of partial fraction decomposition.

a. $\int \frac{1}{x(x^2+1)}\,dx$

b. $\int \frac{x+7}{x^2+2x-8}\,dx$

c. $\int \frac{x^4}{x^2-4}\,dx$

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8. Determine whether each of the following improper integrals is convergent or divergent using the direct comparison test.

a. $\int_1^\infty \frac{2+\sin(x^2)}{\sqrt{x^5+4x}}\,dx$

b. $\int_1^\infty \frac{e^{1/x}}{x+\ln^2 x}\,dx$

c. $\int_0^1 \frac{1}{\sqrt{x}\ln(1+x)}\,dx$

d. $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$

9. Find the values of p for which the following integral converges and evaluate the integral for these values.

$$\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$$

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10. Use the Limit Comparison Test to determine whether the following improper integral converges or diverges:

$$\int_2^\infty \frac{\ln(x)+\sqrt{x}}{x^2-\ln(x)}\,dx$$

Answers

Example questions

1 a. $\frac{1}{3}\arctan\left(\frac{e^x}{3}\right)+C$ b. $\frac{1}{5}\arcsin\left(\frac{5x}{2}\right)+C$

2 $\arcsin\left(\frac{x+1}{2}\right)+C$

3 $\csc x-\cot x+C$

4 $x^2-\frac{3}{2}x+\frac{3}{4}\ln|2x-1|+C$

5 $2\ln\left(2+\sqrt{x}\right)+\frac{4}{2+\sqrt{x}}+C$

6 $-x\cos x+\sin x+C$

7 $x\ln x-x+C$

8 $(x^2-2x+2)e^x+C$

9 $\frac{e^x(\sin x-\cos x)}{2}+C$

10 $\int\cos^n x\,dx=\frac{\cos^{n-1}x\sin x}{n}+\frac{n-1}{n}\int\cos^{n-2}x\,dx$

11 a. $A=\int_0^5 xe^{-x}\,dx$ b. $A=1-6e^{-5}$

12 $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$

13 $\sin x-\frac{1}{3}\sin^3 x+C$

14 $\frac{x}{16}-\frac{\sin(4x)}{64}-\frac{\sin^3(2x)}{48}+C$

15 $-2\sqrt{2}\cos\left(\frac{x}{2}\right)+C$

16 $\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\ln|\sec x|+C$

17 $\frac{1}{2}(\sec x\tan x+\ln|\sec x+\tan x|)+C$

18 $-\frac{1}{8}\cos 4x+\frac{1}{4}\cos 2x+C$

19 $\arctan x+C$

20 $\ln\left(x+\sqrt{a^2+x^2}\right)+C$

21 $-\frac{\sqrt{16-x^2}}{x}-\arcsin\left(\frac{x}{4}\right)+C$

22 $3\ln|x-2|+4\ln|x+3|+C$

23 $\frac{5}{4}\ln\left(x^2+1\right)+\frac{3}{2}\arctan(x)-\frac{5}{2}\ln|x+1|+\frac{4}{x+1}+C$

24 a. $A(b)=1-\frac{1}{b}$ b. Area $=1$

25 It is finite, Area $=\frac{2}{e}$

26 The integral is divergent.

27 The integral is convergent, $I=4$

28 a. Convergent b. Convergent
c. Divergent d. Convergent

29 a. Convergent b. Divergent

Exercises and problems

1
a. $\frac{1}{2}\arcsin(2x)+C$
b. $2\tan x+2\sec x-x+C$
c. $-e^{-\tan x}+C$
d. $\frac{1}{2}\arctan\left(e^{2x}\right)+C$

2
a. $\frac{x^3}{3}\ln(x)-\frac{x^3}{9}+C$
b. $-\frac{2x-3}{2}\cos 2x+\frac{1}{2}\sin 2x+C$
c. $\frac{e^{3x}}{10}(\sin x+3\cos x)+C$
d. $-e^{-x}(x^2+2x+2)+C$

3 Refer to the video.

4 Refer to the video.

5
a. $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+C$
b. $-\frac{4\sqrt{2}}{3}\cos^3\left(\frac{x}{2}\right)+C$
c. $\frac{1}{5}\tan^5 x+\frac{1}{7}\tan^7 x+C$
d. $-x\cos x+\frac{x\cos^3 x}{3}+\frac{2}{3}\sin x+\frac{1}{9}\sin^3 x+C$

6
a. $\operatorname{arcsec}(x)+C$
b. $\arcsin(x)-\sqrt{1-x^2}+C$
c. $\frac{x}{9\sqrt{9-x^2}}+C$
d. $\frac{1}{2}\arctan(2x)+\frac{x}{4x^2+1}+C$

7
a. $\ln|x|-\frac{1}{2}\ln\left(x^2+1\right)+C$
b. $-\frac{1}{2}\ln|x+4|+\frac{3}{2}\ln|x-2|+C$
c. $\frac{x^3}{3}+4x+4\ln\left|\frac{x-2}{x+2}\right|+C$

8
a. Convergent b. Divergent
c. Divergent d. Divergent

9 Converges for $p>1$. Integral value: $\frac{1}{p-1}$

10 The integral is convergent, refer to the video.

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CHAPTER 8

INFINITE SEQUENCES & SERIES - 1

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Motivation

Our main goal in this unit is:

- To learn how to represent complex functions using infinite sums (series).
- To measure how close an approximation (estimation) is to the actual value using "error estimation".
- To make differential equations and integral calculus much easier to solve by using series.

Let's examine the following examples:

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Sequences

Let's express what the concept of a sequence is.

Let's examine the following sequence examples.

- $a_n = 3n + 1$
- $b_n = (-1)^n \cdot n^2$

EXAMPLE 1: Find a formula for the general term a_n of the sequence whose first few terms are given below (assume the pattern continues in the same way):

$$\left\{\frac{5}{3}, -\frac{7}{9}, \frac{9}{27}, -\frac{11}{81}, \frac{13}{243}, \cdots\right\}$$

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Convergence and divergence

DEFINITION: A sequence $\{a_n\}$ is said to converge to the number L if for every number $\epsilon > 0$ there exists an integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

This is denoted by $\lim_{n\to\infty} a_n = L$ or $a_n \to L$, and the number L is called the limit of the sequence. If no such L exists, the sequence $\{a_n\}$ diverges.

Let's show that the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\}$ given by the general term $a_n = \frac{1}{n}$ is convergent.

Let's show that the sequence $\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ given by the general term $a_n = (-1)^{n+1}$ is divergent.

Let's define divergence to infinity or negative infinity for a sequence.

Calculating limits of sequences

Let's examine how we benefit from the methods used in limits when calculating the limits of sequences.

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, where A and B are real numbers, the following rules apply:

Sum Rule: $\lim_{n\to\infty}(a_n + b_n) = A + B$

Difference Rule: $\lim_{n\to\infty}(a_n - b_n) = A - B$

Constant Multiple Rule: $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$, (for any number k)

Product Rule: $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$

Quotient Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, for $B \neq 0$

NOTE: We can use many of the methods used in limit calculation here.

THEOREM: If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $a_n = f(n)$ defined for every integer n is also $\lim_{n\to\infty} a_n = L$.

EXAMPLE 2: Calculate the limits given below.

a. $\lim_{n\to\infty}\left(\frac{3}{n} - 2\right)$

b. $\lim_{n\to\infty} \frac{5n^4-2}{3n^4+n^2}$

c. $\lim_{n\to\infty} \frac{\sin n}{n}$

d. $\lim_{n\to\infty}(-1)^n \frac{1}{n}$

Let's calculate the following limit using L'Hôpital's rule. Let's discuss the condition for using this rule.

$$\lim_{n\to\infty} \frac{\ln n}{n}$$

Recursive definitions

Let's examine recursive definitions through an example.

EXAMPLE 3: The Fibonacci sequence $\{f_n\}$ is given, defined by $f_1 = 1$ and $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for every $n \geq 3$. Write the first 6 terms of the sequence accordingly.

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Bounded monotonic sequences

Let's examine what it means for a sequence to be unbounded or bounded from above or below through examples.

- $a_n = n^2$
- $b_n = \frac{n+1}{n}$

Let's define what it means for a sequence to be monotonic or non-decreasing or non-increasing. Let's evaluate the following examples from this perspective.

- $\{n^3\} = 1, 8, 27, \ldots, n^3, \ldots$
- $\{\frac{n+1}{n}\} = 2, \frac{3}{2}, \frac{4}{3}, \ldots, \frac{n+1}{n}, \ldots$
- $\{\frac{1}{3^n}\} = \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots, \frac{1}{3^n}, \ldots$
- $\{5\} = 5, 5, 5, \ldots, 5, \ldots$
- $\{(-2)^n\} = -2, 4, -8, 16, \ldots$

THEOREM: If a sequence $\{a_n\}$ is both bounded and monotonic, then it is convergent.

Infinite Series

Let's explain the concept of an infinite series. Let's examine an example of an infinite series through partial sums.

Let's define convergence and divergence for a series.

Geometric series

Let's examine the form of geometric series. Let's write an example appropriate for this form. Let's discuss the convergence condition of such a series. For the case of convergence, let's obtain the value to which the series converges.

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

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EXAMPLE 4: Find the sums of the infinite series whose general terms are given below.

a. $\sum_{n=1}^{\infty} 4\left(\frac{1}{3}\right)^{n-1}$ b. $\sum_{n=0}^{\infty} (-1)^n \frac{3}{4^n}$

EXAMPLE 5: A ball released from a height h above the ground rebounds to a height k times the height of the previous fall each time it hits the ground $(0 < k < 1)$. Find the total distance traveled by the ball in terms of h and k.

EXAMPLE 6: Express the repeating decimal 2.3575757... as the sum of an infinite geometric series and write it as a ratio of two integers.

EXAMPLE 7: Find the sum of the "telescoping" series given below:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$$

Divergence test (nth term test)

Let's examine why the following series is divergent.

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots$$

THEOREM: If $\sum_{n=1}^{\infty} a_n$ is convergent, then $a_n \to 0$.

EXAMPLE 8: Determine the convergence or divergence status of the infinite series given below.

a. $\sum_{n=1}^{\infty} (-1)^{n+1}$ b. $\sum_{n=1}^{\infty} \frac{-n}{2n+3}$

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Combining series

When we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to obtain new convergent series.

THEOREM: If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then the following rules apply:

Sum Rule: $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$

Difference Rule: $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$

Constant Multiple Rule: $\sum ka_n = k\sum a_n = kA$ $(k \in \mathbb{R})$.

We can discuss this theorem for divergent series.

EXAMPLE 9: Find the sum of the following series:

$$\sum_{n=1}^{\infty} \frac{2^{n+1}+3}{5^{n-1}}$$

Adding or deleting terms and reindexing

Let us examine how we perform the operations of adding or removing terms and changing indices.

EXAMPLE 10: Examine the convergence of the infinite series given below and find the sum of the series if it converges.

a. $\sum_{n=2}^{\infty} \frac{1}{4^n}$ b. $\sum_{n=2}^{\infty} \frac{2^{n+1}}{3^n}$

Integral Test

Let us use the following theorem to show the divergence of the harmonic series.

THEOREM: Let $\sum_{n=1}^{\infty} a_n$ be a series consisting of non-negative terms. This series is convergent if and only if its partial sums are bounded from above.

EXAMPLE 11: Let us examine the convergence of the series given below:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

THEOREM: Let $\{a_n\}$ be a sequence with positive terms. Suppose that for all $x \geq N$ $(N \in \mathbb{Z}^+)$, the function f is a continuous, positive, and decreasing function and satisfies $a_n = f(n)$. In this case, the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ either both converge or both diverge.

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Let us examine the convergence and divergence condition of the p-series:

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$$

EXAMPLE 12: Determine the convergence or divergence of the infinite series given below.

a. $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ b. $\sum_{n=2}^{\infty}\frac{1}{n\ln^2(n)}$

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Error estimation with the integral test

Suppose that a series $\sum a_n$ with positive terms is shown to be convergent by the integral test. In this case, let us estimate the magnitude of the remainder term R_n, which measures the difference between the sum of the series S and the n-th partial sum s_n.

EXAMPLE 13: The sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ will be estimated using the error estimation method with the integral test by taking $n=4$. Determine the lower and upper bounds for the sum of the series S. (Compare the result you obtained with the value $\frac{\pi^2}{6}\approx 1.64493$.)

Comparison Tests

Direct comparison test

THEOREM: Let $\sum a_n$ and $\sum b_n$ be two series satisfying the condition $0\le a_n\le b_n$ for every n. In this case:

I. If $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.

II. If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

EXAMPLE 14: Examine the convergence of the infinite series given below using the direct comparison test.

a. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ b. $\sum_{n=1}^{\infty} \frac{1}{2^n+\sqrt{n}}$

EXAMPLE 15: Examine the convergence of the infinite series given below using the limit comparison test.

a. $\sum_{n=1}^{\infty} \frac{3n^2+5n}{n^4+2n-1}$ c. $\sum_{n=1}^{\infty} \frac{n+1}{n\ln(n)}$

b. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$

Limit comparison test

THEOREM: Let $a_n > 0$ and $b_n > 0$ for every $n \geq N$ ($N \in \mathbb{Z}$).

I. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

II. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

III. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

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Absolute Convergence; The Ratio and Root Tests

DEFINITION: If the series $\sum |a_n|$ converges, then $\sum a_n$ converges absolutely.

THEOREM: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

EXAMPLE 16: Determine whether the series given below are absolutely convergent and investigate their convergence status.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$ b. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

EXAMPLE 17: Investigate the convergence status of the series given below using the ratio test.

a. $\sum_{n=1}^{\infty} \frac{5^n}{n!}$ c. $\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$

b. $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

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Ratio test

THEOREM: $\sum a_n$ be any series and suppose that:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

In this case;

I. If $\rho < 1$, the series is absolutely convergent.

II. If $\rho > 1$ or ρ is infinite, the series is divergent.

III. If $\rho = 1$, the test is inconclusive.

Root test

THEOREM: Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$

. In this case;

I. If $\rho < 1$, the series is absolutely convergent.

II. If $\rho > 1$ or ρ is infinite, the series is divergent.

III. If $\rho = 1$, the test is inconclusive.

EXAMPLE 18: Investigate the convergence status of the series given below using the root test.

a. $\sum_{n=1}^{\infty}\left(\frac{3n+1}{4n-2}\right)^n$ b. $\sum_{n=1}^{\infty}\frac{2^n}{n^n}$

Alternating Series and Conditional Convergence

THEOREM:

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

A series of this form converges if the following conditions are met:

I. All u_n values must be positive.

II. The sequence u_n must satisfy the condition $u_n \geq u_{n+1}$ for every integer $n \geq N$ sufficiently large, i.e., it must exhibit a monotonically decreasing character.

III. $u_n \to 0$ must hold.

EXAMPLE 19: Investigate the convergence of the infinite series given below:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Let's examine the *alternating series estimation theorem* for the following infinite series, which is a variation of the harmonic series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Conditional convergence

DEFINITION: A series that is convergent but not absolutely convergent is called a conditionally convergent series.

Let's examine the following series for conditional convergence:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

EXAMPLE 20: Investigate the convergence status of the series given below.

a. $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ c. $\sum_{n=1}^{\infty}(-1)^n\frac{\arctan n}{n^2+1}$

b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+5}$

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Exercises and Problems

1. Investigate the convergence status of the sequences whose general terms are given below. If convergent, find the limit.

a. $a_n = \frac{3n^2-5n+1}{2n^2+7}$

b. $a_n = \left(1+\frac{5}{n}\right)^n$

c. $a_n = \frac{\ln\left(n^2\right)}{n}$

d. $a_n = \frac{(-1)^n \sin(n)}{n^2+1}$

e. $a_n = \frac{n!}{n^n}$

f. $a_n = \frac{n^2+\ln n}{3n+2}$

2. Determine the monotonicity and boundedness of the sequences whose general terms are given below.

a. $a_n = \frac{4n-3}{2n+5}$

b. $a_n = \frac{(3n+1)!}{(n+2)!}$

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3. Investigate the convergence status of the infinite series given below and find the sum of the series if it converges.

a. $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^{n+1}}{5^n}\right)$

b. $\sum_{n=2}^{\infty}\left(1-\frac{3}{2^n}\right)$

c. $\sum_{n=0}^{\infty}\frac{\cos n\pi}{2^n}$

d. $\sum_{n=1}^{\infty}\frac{5}{n(n+1)}$

e. $\sum_{n=1}^{\infty}\left(\frac{1}{2^{1/n}}-\frac{1}{2^{1/(n+1)}}\right)$

f. $\sum_{n=1}^{\infty}\ln\frac{1}{2^n}$

4. Apply the divergence test to the infinite series given below and state the result obtained.

a. $\sum_{n=0}^{\infty}\frac{e^n}{e^n+n^2}$

b. $\sum_{n=1}^{\infty}\frac{n(n+1)}{n^3}$

5. For each of the following geometric series, find the values of x for which the series converges. Also, express the sum of the series as a function of x for these x values.

a. $\sum_{n=0}^{\infty}3^n x^n$

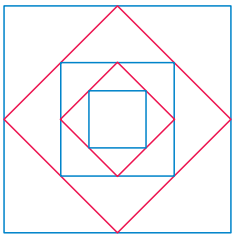
b. $\sum_{n=0}^{\infty}\left(\frac{x+2}{2}\right)^n$

c. $\sum_{n=0}^{\infty}\cos^n x$

d. $\sum_{n=0}^{\infty}\left(\frac{\ln x}{2}\right)^n$

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6. Starting with a square of side length a, each new square is formed by joining the midpoints of the previous square. Find the total sum of the perimeters of all squares formed when this process is repeated infinitely, in terms of a.



7. Determine the convergence or divergence status of the infinite series given below using the integral test.

a. $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$

b. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

8. Determine the convergence status of the series given below using comparison tests:

a. $\sum_{n=1}^{\infty} \frac{1}{2n^2+n}$

b. $\sum_{n=2}^{\infty} \frac{n+5}{n^2-n}$

c. $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}e^n}$

d. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

9. Investigate the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n!}$. (Hint: First show that $1/n! \leq 1/n(n-1)$ for $n \geq 2$.)

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10. Investigate the convergence status of the series given below using the ratio or root test.

a. $\sum_{n=1}^{\infty} \frac{(2n)!}{3^n(n!)^2}$ b. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

Answers

Example questions

1	$a_n = (-1)^{n+1}\frac{2n+3}{3^n}$
2	a. -2 b. $\frac{5}{3}$ c. 0 d. 0
3	{1,1,2,3,5,8}
4	a. 6 b. $\frac{12}{5}$
5	$S = h\left(\frac{1+k}{1-k}\right)$
6	$\frac{389}{165}$
7	$\frac{1}{2}$
8	a. Divergent b. Divergent
9	$\frac{125}{12}$
10	a. Convergent, $\frac{1}{12}$ b. Convergent, $\frac{8}{3}$
11	Convergent
12	a. Convergent b. Convergent
13	$1.62361 \le S \le 1.67361$
14	a. Divergent b. Convergent
15	a. Convergent b. Convergent c. Divergent
16	a. The series is absolutely convergent; therefore it is convergent. b. The series is absolutely convergent; therefore it is convergent.
17	a. Convergent b. Divergent c. Ratio test is inconclusive ($L = 1$)
18	a. Convergent b. Convergent
19	Convergent
20	a. Conditionally convergent b. Divergent c. Absolutely conver

Exercises and problems

1	a. Convergent, $L = \frac{3}{2}$ b. Convergent, $L = e^5$ c. Convergent, $L = 0$ d. Convergent, $L = 0$ e. Convergent, $L = 0$ f. Divergent, $+\infty$
2	a. Monotonically non-decreasing and bounded. Lower bound $\frac{1}{7}$, up b. Monotonically non-decreasing, bounded below by 4, unbounded
3	a. Convergent, $\frac{7}{6}$ b. Divergent c. Convergent, $\frac{2}{3}$ d. Convergent, 5 e. Convergent, $-\frac{1}{2}$ f. Divergent
4	a. Divergent b. $\lim_{n\to\infty} a_n = 0$, test is inconclusive
5	a. $x \in (-\frac{1}{3},\frac{1}{3})$, $S(x) = \frac{1}{1-3x}$ b. $x \in (-4,0)$, $S(x) = -\frac{2}{x}$ c. $x \neq k\pi$ $(k \in \mathbb{Z})$, $S(x) = \frac{1}{1-\cos x}$ d. $x \in (e^{-2},e^2)$, $S(x) = \frac{2}{2-\ln x}$
6	$L = 4a(2+\sqrt{2})$
7	a. Divergent b. Divergent
8	a. Convergent b. Divergent c. Convergent d. Divergent
9	Convergent
10	a. Divergent b. Convergent

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CHAPTER 8

INFINITE SEQUENCES & SERIES - 2

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Power Series

DEFINITION: $x = 0$ centered power series is a series of the following form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

$x = a$ centered power series is a series of the following form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

Here, a is the center and $c_0, c_1, c_2, \cdots, c_n, \cdots$ are constant coefficients.

Let us examine the following geometric power series and obtain the interval on which the series converges. For this interval, let us find the function to which the series converges.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

Let us examine the following geometric power series and obtain the interval on which the series converges. For this interval, let us find the function to which the series converges.

$$1 - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-1)^n + \cdots$$

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EXAMPLE 1: Determine the interval of convergence for the power series given below.

a. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

b. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

c. $\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \cdots$

THEOREM: The convergence behavior of a power series $\sum_{n=0}^{\infty} a_n x^n$ is defined in terms of the behavior of the series at specific points as follows:

I. **Convergence:** If the series converges at $x = c$ $(c \neq 0)$, then for all values of x closer to the center than the point c, that is, for every $|x| < |c|$, the series converges absolutely.

II. **Divergence:** If the series diverges at $x = d$, then for all values of x farther from the center than the point d, that is, for every $|x| > |d|$, the series diverges.

NOTE: This theorem tells us that power series always have a convergence interval symmetric about the center.

Let us discuss the concept of the *radius of convergence* through the power series $\sum a_n(x-a)^n$.

Operations on Power Series

Let us examine the *series multiplication* operation for power series through the following example:

$$\left(\sum_{n=0}^{\infty}(-x)^n\right)\cdot\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\right)$$

In a power series, a function $f(x)$ can be substituted in place of the variable x.

Let us examine this on the following series for $f(x) = 4x^2$, together with its results:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

A power series can be differentiated term by term.

Let us examine this on the following series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

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A power series can be integrated term by term.

Let us examine this on the following series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

EXAMPLE 2: Determine the function $f(x)$ defined by the following power series:

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \quad -1 \le x \le 1.$$

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EXAMPLE 3: The following series is given for the function $\sin x$ and converges to $\sin x$ for all values of x:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

a. Find the first few terms of the series for $\cos x$. For which values of x does the series converge?

b. By substituting $2x$ for x in the series for $\sin x$, find a series that converges to $\sin 2x$ for all values of x.

c. Using series multiplication, compute the first terms of the series for $2 \sin x \cos x$. Compare your answer with the answer in part (b).

Taylor and Maclaurin Series

Suppose that a function $f(x)$ is the sum of a power series with a positive radius of convergence around $x = a$. That is, let us write the function in the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

Now let us obtain the coefficients here in terms of the derivatives of the function. Using these coefficients, let us define the Taylor and Maclaurin series.

EXAMPLE 4: Find the Taylor series of the function $f(x) = \ln x$ about the point $a = 1$ and determine the radius of convergence of this series.

f has derivatives of all orders at the point $x = a$. In this case, the Taylor polynomial of degree n, denoted by $P_n(x)$, representing the function f near the point a, is defined as follows:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Let us find the Taylor series generated at the point $x = 0$ for the function $f(x) = e^x$. Then, using the first three terms of this series, let us write the Taylor polynomials $P_0(x)$, $P_1(x)$, and $P_2(x)$.

EXAMPLE 5: Find the Maclaurin series expansion of the function $f(x) = \sin x$.

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EXAMPLE 6: Find the Taylor expansion of the function $f(x) = x^3 - 5x + 1$ at the point $x = 1$.

Error estimation

Let us examine error estimates for Taylor series through examples.

EXAMPLE 7: Estimate the error that occurs when the polynomial $P_2(x) = 1 - \frac{x^2}{2}$ is used to approximate $\cos x$ at $x = 0.2$.

EXAMPLE 8: Approximate the value of $1/e$ so that the error is less than $1/100$.

THEOREM: If the function f and its first n derivatives $f', f'', \cdots, f^{(n)}$ are continuous on the closed interval between a and b, and if $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that the following equality holds:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

Let us show that the Taylor series of the function $\cos x$ at the point $x = 0$ converges to $\cos x$ for every value of x.

EXAMPLE 9: When values of x are small, the approximation $e^x \approx 1 + x + \frac{x^2}{2}$ is used. Estimate the error of this approximation when $|x| < 0.1$.

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Applications of Taylor Series

Binomial series

$(1+x)^m$ has the following infinite series expansion for $|x|<1$:

$$(1+x)^m=\sum_{k=0}^{\infty}\binom{m}{k}x^k=1+mx+\frac{m(m-1)}{2!}x^2+\cdots$$

EXAMPLE 10: The function $f(x)=\sqrt{1+x}$ is given.

a. Write the first three terms of the Binomial series expansion of this function centered at $x=0$.

b. Similarly, write the first three terms of the Binomial series expansion of the function $g(x)=\sqrt{1-x^2}$ centered at $x=0$.

Evaluation of non-elementary integrals

Let us examine through an example how Taylor series are used in evaluating integrals.

EXAMPLE 11: Using the Maclaurin series of the function $\cos x^3$, answer the following:

a. Express the indefinite integral $\int \cos x^3\,dx$ as a power series.

b. Estimate the value of the definite integral $\int_0^1 \cos x^3\,dx$ with an error less than 10^{-2}.

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Arctangents

Let us examine through an example how Taylor series are used in arctangent calculations.

EXAMPLE 12: Using the Taylor series of the function $f(x)=\arctan x$ about $x=0$, suppose that we want to compute the value $\frac{\pi}{4}$ with an error smaller than 10^{-2}.

a. Write the Taylor series expansion of the function $\arctan x$.

b. Determine how many terms of the series must be summed to compute $\frac{\pi}{4}$ with the desired accuracy.

Evaluation of indeterminate forms

Sometimes we can evaluate indeterminate forms by expressing the relevant functions in terms of Taylor series. Let us examine the examples.

EXAMPLE 13: Evaluate the following limit using the Taylor series:

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}$$

EXAMPLE 14: Evaluate the following limit using the Taylor series:

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}$$

Euler's identity

Let us write Euler's identity and show this identity using Taylor series.

Important Maclaurin series and their radii of convergence:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots,\quad R=1$$

$$\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}+\cdots,\quad R=1$$

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots,\quad R=1$$

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\cdots,\quad R=1$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots,\quad R=\infty$$

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots,\quad R=\infty$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots,\quad R=\infty$$

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Exercises and Problems

1. For the power series given below:

i. Find the radius of convergence and the interval of convergence.

ii. For which values of x does the series converge absolutely?

iii. For which values of x does the series converge conditionally?

a. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 4^n}$

b. $\sum_{n=0}^{\infty} \frac{(-1)^n (2x+1)^n}{\sqrt{n+1}}$

c. $\sum_{n=1}^{\infty} \frac{x^n}{n^2 5^n}$

2. Find the interval of convergence of the series given below and express its sum as a function of x on this interval:

$$\sum_{n=0}^{\infty} (e^x - 1)^n$$

3. Find the sum of the series given below:

$$\frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \cdots + (-1)^{n-1} \frac{1}{n \cdot 2^n} + \cdots$$

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4. Find the Taylor series (Maclaurin series) of the following functions at the point $x = 0$:

a. $f(x) = \frac{1}{1+x^3}$ b. $g(x) = e^{-x^2}$

5. Let the function $f(x) = \frac{3x^2}{3-x}$ be given.

a. Find the Maclaurin series (Taylor series about 0) of the function $f(x)$.

b. Find the interval of convergence of this series.

6. Let $G(x) = \int_0^x \sin t^2 \, dt$ be given.

a. Find the Maclaurin series (Taylor series about 0) of the function $G(x)$. (You may use the Maclaurin series of known functions.)

b. Approximate the value of $G(1)$ so that the error is less than 0.001.

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Answers

Example questions

1 a. $[-1,1]$ b. $(-\infty,\infty)$ c. $\{0\}$

2 $f(x)=\arctan x$

3 Go to the video.

4 a. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x-1)^n$ b. $R=1$

5 $\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$

6 $f(x)=-3-2(x-1)+3(x-1)^2+(x-1)^3$

7 Error $\leq\frac{(0.2)^4}{24}$

8 $\frac{3}{8}$

9 Error $\leq\frac{e^{0.1}}{6000}$

10 a. $1+\frac{1}{2}x-\frac{1}{8}x^2$ b. $1-\frac{1}{2}x^2-\frac{1}{8}x^4$

11 a. $C+x-\frac{x^7}{7\cdot 2!}+\frac{x^{13}}{13\cdot 4!}-\cdots$ b. ≈ 0.929

12 a. $\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$

b. $n=50$ terms (last term $\frac{1}{99}$)

13 $\frac{1}{2}$

14 $\frac{1}{6}$

Exercises and problems

1 a. i. $R=4,[-1,7)$ ii. $(-1,7)$ iii. $x=-1$

b. i. $R=1/2,(-1,0]$ ii. $(-1,0)$ iii. $x=0$

c. i. $R=5,[-5,5]$ ii. $[-5,5]$ iii. $\emptyset$

2 $(-\infty,\ln 2)$, $f(x)=\frac{1}{2-e^x}$

3 Sum $=\ln\left(\frac{3}{2}\right)$

4 a. $\sum_{n=0}^{\infty}(-1)^n x^{3n}$ b. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$

5 a. $\sum_{n=0}^{\infty}\frac{x^{n+2}}{3^n}$

b. $(-3,3)$

6 a. $G(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$

b. $G(1)\approx\frac{13}{42}\approx 0.3095$

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CHAPTER 9

PARAMETRIC EQUATIONS AND POLAR COORDINATES

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Parametrization of Plane Curves

DEFINITION: As the variable t takes values in a certain interval, the set of points (x, y) determined by the equations $x = f(t)$ and $y = g(t)$ is called a parametric curve. These equations are called the parametric equations of the curve.

EXAMPLE 1: Considering the parabolic curve given by the equation $y = x^2$, answer the following questions:

a. Using the parameter $x = t$, write the parametric equations of the curve for $t \in (-\infty, \infty)$.

b. Using the parametrization $x = \sqrt{t}$, write the parametric equations of the curve for $t \geq 0$ and state which part of the curve this parametrization represents.

EXAMPLE 2: Sketch the curve defined by the parametric equations given below:

$$x = t^2 - 1, \quad y = t + 2, \quad -2 \leq t \leq 2$$

EXAMPLE 3: Sketch the curve defined by the parametric equations given below:

$$x = \sin t, \quad y = \cos t, \quad 0 \leq t \leq 2\pi$$

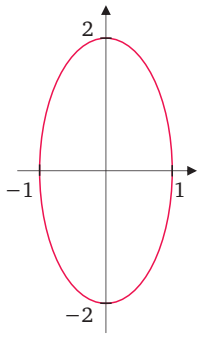
EXAMPLE 4: Match the parametric equations given below with the numbered parametric curves.

a. $x = t, \quad y = t\cos t^2$

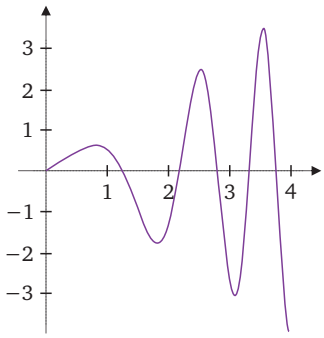
b. $x = t, \quad y = \sqrt{1 - t^2}$

c. $x = \sin t, \quad y = 2\cos t$

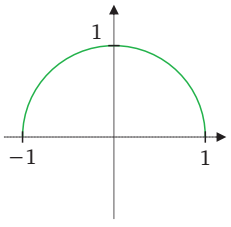
d. $x = t\sin t, \quad y = t\cos t$



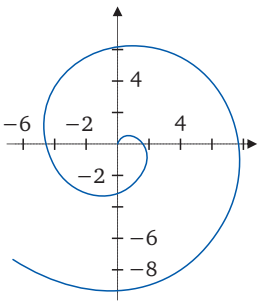
I



II



III



IV

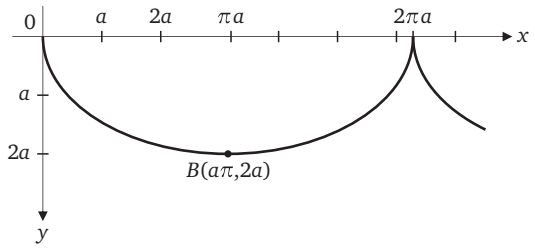
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Cycloid

Let us examine the cycloid curve through an example.

EXAMPLE 5: A wheel of radius a rolls without slipping along a horizontal line. Obtain the parametric equations of the path traced by a point P on the wheel that is initially located directly below the center of the wheel. The point P is initially at the origin.

Let us examine the brachistochrone and tautochrone curves.



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EXAMPLE 6: Find the equation of the tangent line to the ellipse given by the equations $x = 2\cos t$ and $y = 3\sin t$ at the point corresponding to the parameter value $t = \pi/6$.

EXAMPLE 7: For the parametric curve given by $x = t^2 + 1$ and $y = t^3 + t$, find the expression for $\frac{d^2y}{dx^2}$ in terms of t.

Calculus with Parametric Curves

Tangent lines and areas

Let us derive the parametric formulas for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

EXAMPLE 8: Using an integral, calculate the area of the region enclosed by the curve with parametric equations $x = 2\cos t$ and $y = 2\sin t$ $(0 \leq t \leq \pi/2)$ and the axes.

Length of a parametrically defined curve

Let us examine how to find the length of a parametrically defined curve.

EXAMPLE 9: Calculate the length of the curve $x = e^t \cos t$, $y = e^t \sin t$ on the interval $0 \leq t \leq \pi$.

Surfaces of Revolution

Let us examine how to compute the surface area generated by revolving a parametric curve.

EXAMPLE 10: Calculate the surface area generated by revolving the portion of the curve given by the parametric equations $x = \cos^3 t$ and $y = \sin^3 t$ for $0 \leq t \leq \pi/2$ about the x-axis.

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Polar Coordinates

Let us define polar coordinates.

EXAMPLE 11: Find all polar coordinates of the point $P(3, \pi/4)$.

Let us draw the graphs corresponding to the polar equations.

a. $r = 1$ ve $r = -1$

b. $\theta = \pi/4$, $\theta = 5\pi/4$ ve $\theta = -\pi/4$

c. $2 \leq r \leq 4$ ve $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

d. $-2 \leq r \leq 1$ ve $\theta = \frac{\pi}{6}$

e. $\frac{3\pi}{4} \leq \theta \leq \pi$

Let us derive the transformations between polar coordinates and Cartesian coordinates.

EXAMPLE 12: Find the polar equation of the circle given by the equation $(x-2)^2 + y^2 = 4$.

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EXAMPLE 13: Convert the polar equations given below into equivalent Cartesian equations and describe their graphs.

a. $r \sin\theta = 5$

b. $r^2 = -6r \sin\theta$

c. $r = \frac{6}{3\cos\theta + 2\sin\theta}$

Graphing Polar Coordinate Equations

Let us examine the symmetry tests in polar coordinates.

I. Symmetry with respect to the x-axis: If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ also lies on the graph.

II. Symmetry with respect to the y-axis: If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ also lies on the graph.

III. Symmetry with respect to the origin: If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ also lies on the graph.

Let us derive the slope of a polar curve given in the form $r = f(\theta)$.

If the curve $r = f(\theta)$ passes through the origin, let us determine its slope at that point.

EXAMPLE 14: Draw the curve given by the equation $r = 1 - \cos\theta$ in the Cartesian xy-plane.

Areas and Lengths in Polar Coordinates

Let us derive the differential area element required for area calculations.

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EXAMPLE 15: Find the area of the region in the xy-plane bounded by the cardioid $r = 2(1 + \sin\theta)$.

Let us write a general integral for the area of the region satisfying $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$ and $\alpha \leq \theta \leq \beta$.

EXAMPLE 16: In the polar coordinate system, consider the region bounded by the cardioid $r_1(\theta) = 2 + 2\cos\theta$ and the circle $r_2(\theta) = 2$. Compute the area of the region that lies outside the circle but inside the cardioid.

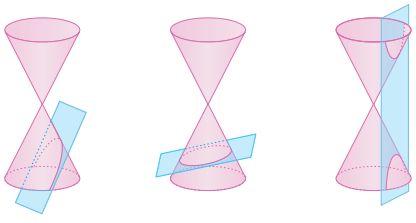
Let us examine how to determine the length of a polar curve.

EXAMPLE 17: Calculate the total length of the cardioid given by the equation $r = 1 + \cos\theta$.

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Conic Sections

Let us define the parabola, ellipse, and hyperbola geometrically and examine their shapes.



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Parabola

DEFINITION: In a plane, the set of points that are equidistant from a fixed point and a fixed line is called a parabola. This fixed point is called the focus of the parabola, and the fixed line is called the directrix.

EXAMPLE 18: Find the focus and the directrix equation of the parabola given by $x^2 = -12y$.

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Ellipse

DEFINITION: An ellipse is the set of points in the plane for which the sum of the distances to two fixed points is constant. These two fixed points are called the foci of the ellipse. The line passing through the foci is called the focal axis, and the midpoint between the foci on this axis is called the center. The points where the focal axis intersects the ellipse are called the vertices.

EXAMPLE 19: For the ellipse given by the equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$, calculate the following quantities:

a. Find the lengths of the semi-major axis (a) and the semi-minor axis (b).

b. Compute the distance from the center to a focus (c).

c. Write the coordinates of the foci of the ellipse.

d. Write the coordinates of the vertices of the ellipse.

e. State the center of the ellipse.

Hyperbola

DEFINITION: A hyperbola is the set of points in the plane for which the absolute difference of the distances to two fixed points (the foci) is constant. The line passing through the foci is called the focal axis. The midpoint between the foci is the center of the hyperbola. The points where the focal axis intersects the hyperbola are called the vertices.

EXAMPLE 20: Find the following characteristic properties of the hyperbola given by $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

a. Center
b. Vertices
c. Foci
d. Asymptotes

Let us discuss what happens if x and y are interchanged in the equations we wrote for the parabola, ellipse, and hyperbola.

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Conics in Polar Coordinates

Eccentricity

In conic sections, the eccentricity is denoted by e.

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$

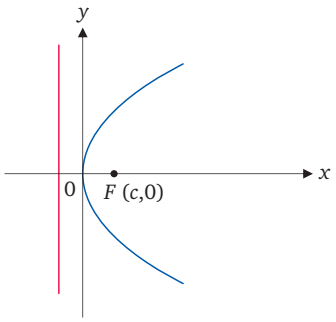
$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a},$$

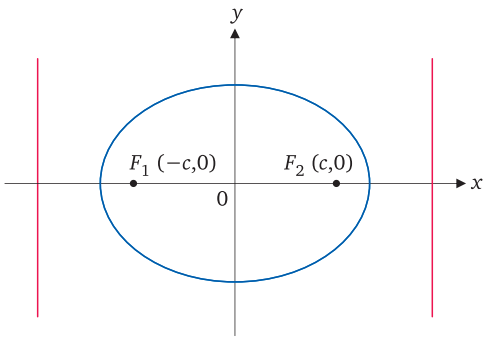
for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

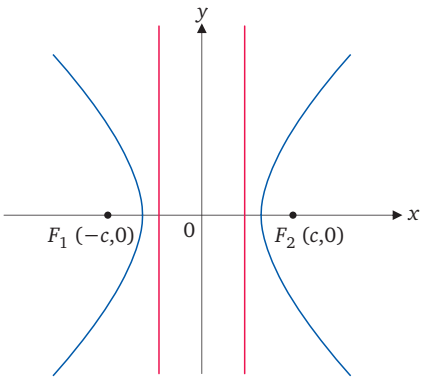
$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a},$$

and for the parabola $e = 1$.

Let us examine the value of eccentricity in detail using the following figures.







EXAMPLE 21: Find the Cartesian equation of the hyperbola whose center is at the origin, one focus is at $(5,0)$, and the corresponding directrix is $x = \frac{9}{5}$.

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Polar equations

Let us see how to derive the polar equation of a parabola, ellipse, or hyperbola.

EXAMPLE 22: Find the standard equation of the hyperbola whose eccentricity is $e = 2$ and whose directrix is the line $x = 3$. The focus of the hyperbola is at the origin.

EXAMPLE 23: Consider the curve whose equation is $r = \frac{12}{3-6\cos\theta}$.

a. Find the eccentricity (e) of this conic section and determine its type.

b. Find the equation of the directrix.

c. Determine the vertices of the curve on the x-axis.

Let us derive the polar equation of an ellipse with eccentricity e and semi-major axis a.

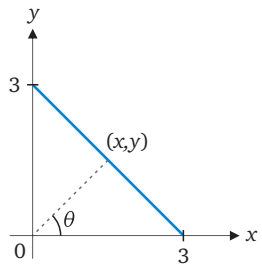
Let us derive the equation of a line in polar coordinates.

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Let us derive the equation of a circle in polar coordinates.

Exercises and Problems

1. Parametrize the line segment joining the points $(0,3)$ and $(3,0)$ using the angle θ from the origin as the parameter, as shown in the figure.



2. For the circle $(x+3)^2+(y-1)^2=4$, find a parametrization that starts at the point $(-1,1)$, moves clockwise around the circle for one full revolution, and uses the central angle θ as the parameter.

3. Draw the curve given by the equation $r^2=\sin\theta$ in the Cartesian xy-plane.

4. Compute the area of the region of the circle given by the polar equation $r=4\cos\theta$ for $\pi/6\leq\theta\leq\pi/3$.

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5. Show that the equation $x^2-4y^2-4x+8y-4=0$ represents a hyperbola. Find the center, the foci, and the asymptotes of this hyperbola.

6. For a conic section with eccentricity $e=\frac{3}{5}$ whose one focus is at the origin of the polar coordinate plane, the corresponding directrix is given by $r\cos\theta=10$. Find a polar equation for this conic section.

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Answers

Example problems

1	a. $x=t,\quad y=t^2,\quad t\in\mathbb{R}$ b. $x=\sqrt{t},\quad y=t,\quad x\geq 0$ (Right branch)
2	See the video.
3	See the video.
4	a. II b. III c. I d. IV
5	$x=a(\theta-\sin\theta)\quad y=a(1-\cos\theta)$
6	$y=-\frac{3\sqrt{3}}{2}x+6$
7	$\frac{d^2y}{dx^2}=\frac{3t^2-1}{4t^3}$
8	$A=\pi$
9	$L=\sqrt{2}(e^{\pi}-1)$
10	$S=\frac{6\pi}{5}$
11	$(3,\frac{\pi}{4}+2n\pi)$ ve $(-3,\frac{5\pi}{4}+2n\pi),\ n\in\mathbb{Z}$
12	$r=4\cos\theta$
13	a. $y=5$, line b. $x^2+(y+3)^2=9$, circle c. $3x+2y=6$, line
14	See the video.
15	$A=6\pi$
16	$A=8+\pi$
17	$L=8$
18	Focus: $(0,-3)$ Directrix: $y=3$
19	a. $a=5,\ b=3$ b. $c=4$ c. (4,0) and (−4,0) d. (5,0) and (−5,0) e. (0,0)
20	a. $M(0,0)$ b. $V_1(-3,0),\ V_2(3,0)$ c. $F_1(-5,0),\ F_2(5,0)$ d. $y=\pm\frac{4}{3}x$
21	$\frac{x^2}{9}-\frac{y^2}{16}=1$
22	$r=\frac{6}{1+2\cos\theta}$

23	a. $e = 2$, Hyperbola b. $x = -2$ c. $(-4,0)$ and $(-\frac{4}{3},0)$

Exercises and problems

1	$x = \frac{3\cos\theta}{\cos\theta+\sin\theta}$, $y = \frac{3\sin\theta}{\cos\theta+\sin\theta}$, $0 \le \theta \le \frac{\pi}{2}$
2	$x = -3 + 2\cos\theta$, $y = 1 - 2\sin\theta$, $0 \le \theta \le 2\pi$
3	See the video.
4	$A = \frac{2\pi}{3}$
5	Center: $(2,1)$ Foci: $(2 \pm \sqrt{5},1)$ Asymptotes: $y - 1 = \pm\frac{1}{2}(x-2)$
6	$r = \frac{30}{5+3\cos\theta}$

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KALKÜLÜS 1–2

TÜRKÇE-İNGİLİZCE TERİM KARŞILIKLARI

1. Ünite: Limitler ve Süreklilik

Limit Limit

Limit kuralları Limit laws

Ortalama değişim oranı Average rate of change

Anlık değişim oranı Instantaneous rate of change

Kiriş doğrusu Secant line

Teğet doğrusu Tangent line

Sıkıştırma teoremi Sandwich theorem

Limitin tam tanımı Precise definition of a limit

Tek taraflı limit One-sided limit

Sağdan limit Right-hand limit

Soldan limit Left-hand limit

Sonsuzdaki limit Limit at infinity

Sonsuz limit Infinite limit

Yatay asimptot Horizontal asymptote

Düşey asimptot Vertical asymptote

Eğik asimptot Oblique asymptote

Süreklilik Continuity

Sürekli fonksiyon Continuous function

Süreksizlik Discontinuity

Kaldırılabilir süreksizlik Removable discontinuity

Sıçrama süreksizliği Jump discontinuity

Sonsuz süreksizlik Infinite discontinuity

Salınımlı süreksizlik Oscillating discontinuity

Sürekli genişleme Continuous extension

Ara değer teoremi Intermediate value theorem

Tam değer fonksiyonu Greatest integer function

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2. Ünite: Türev

Türev Derivative

Türev alma Differentiation

Türevlenebilir Differentiable

Fark bölümü Difference quotient

Bir noktadaki eğim Slope at a point

Normal doğrusu Normal line

Köşe Corner

Sivri uç Cusp

Kuvvet kuralı Power rule

Çarpım kuralı Product rule

Bölüm kuralı Quotient rule

Yüksek mertebeden türev Higher-order derivative

Değişim oranı Rate of change

Konum Position

Yer değiştirme Displacement

Hız Velocity

Sürat Speed

İvme Acceleration

Silkme Jerk

Zincir kuralı Chain rule

Kapalı türev Implicit differentiation

Ters trigonometrik fonksiyon Inverse trigonometric function

İlişkili oranlar Related rates

Doğrusallaştırma Linearization

Diferansiyel Differential

3. Ünite: Türev Uygulamaları

Mutlak (global) maksimum Absolute (global) maximum

Mutlak (global) minimum Absolute (global) minimum

Yerel maksimum Local maximum

Yerel minimum Local minimum

Ekstremum değer Extreme value

Uç değer teoremi Extreme value theorem

Kritik nokta Critical point

Rolle teoremi Rolle's theorem

Ortalama değer teoremi Mean value theorem

Artan fonksiyon Increasing function

Azalan fonksiyon Decreasing function

Birinci türev testi First derivative test

İkinci türev testi Second derivative test

Konkavlık Concavity

Yukarı konkav Concave up

Aşağı konkav Concave down

Dönüm noktası Point of inflection

Belirsiz biçim Indeterminate form

L'Hôpital kuralı L'Hôpital's rule

Optimizasyon Optimization

Newton yöntemi Newton's method

Ters türev Antiderivative

4. Ünite: İntegral

Alan Area

Sonlu toplam Finite sum

Alt toplam Lower sum

Üst toplam Upper sum

Sigma gösterimi Sigma notation

Bölüntü Partition

Bölüntünün normu Norm of a partition

Riemann toplamı Riemann sum

İntegrallenebilir Integrable

İntegral sınırları Limits of integration

Bir fonksiyonun ortalama değeri Average value of a function

Kalkülüsün temel teoremi Fundamental theorem of calculus

Toplam alan Total area

Eğriler arasındaki alan Area between curves

5. Ünite: İntegrasyon Uygulamaları

Hacim Volume

Kesit alanı Cross-sectional area

Dilimleme yöntemi Method of slicing

Dönel cisim Solid of revolution

Dönme ekseni Axis of revolution

Disk yöntemi Disk method

Pul yöntemi Washer method

Silindirik kabuk yöntemi Shell method

Silindirik kabuk Cylindrical shell

Eğri uzunluğu Arc length

Yay uzunluğu diferansiyeli Arc length differential

Düzgün (pürüzsüz) eğri Smooth curve

Dönel yüzey Surface of revolution

Yüzey alanı Surface area

İş Work

Yay sabiti (katsayısı) Spring constant

Hooke yasası Hooke's law

Akışkan basıncı Fluid pressure

Akışkan kuvveti Fluid force

Ağırlık yoğunluğu Weight density

Moment Moment

Kütle merkezi Center of mass

Ağırlık merkezi (geometrik merkez) Centroid

Yoğunluk Density

İnce levha Thin plate

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6. Ünite: İntegraller ve Aşkın Fonksiyonlar

Aşkın fonksiyon Transcendental function

Doğal logaritma Natural logarithm

Üstel fonksiyon Exponential function

Logaritmik türev alma Logarithmic differentiation

Ters fonksiyon Inverse function

Üstel büyüme Exponential growth

Üstel azalma (azalış) Exponential decay

Yarılanma ömrü (yarı ömür) Half-life

Diferansiyel denklem Differential equation

Ayrılabilir diferansiyel denklem Separable differential equation

Başlangıç değer problemi Initial value problem

Newton soğuma yasası Newton's law of cooling

Hiperbolik fonksiyon Hyperbolic function

Hiperbolik sinüs Hyperbolic sine

Hiperbolik kosinüs Hyperbolic cosine

Ters hiperbolik fonksiyon Inverse hyperbolic function

Zincir eğrisi (katener) Catenary

Göreli büyüme oranı Relative rate of growth

Daha hızlı büyümek Grow faster

Aynı hızda büyümek Grow at the same rate

7. Ünite: İntegrasyon Teknikleri

Belirsiz integral Indefinite integral

Belirli integral Definite integral

İntegrand (tümlevlenen) Integrand

Değişken değiştirme Substitution

Kısmi integrasyon Integration by parts

Tablo yöntemi Tabular integration

İndirgeme formülü Reduction formula

Trigonometrik integral Trigonometric integral

Trigonometrik değişken değiştirme Trigonometric substitution

Kısmi kesirler Partial fractions

Kısmi kesirlere ayrıştırma Partial fraction decomposition

Rasyonel fonksiyon Rational function

Basit (has) kesir Proper fraction

İndirgenemez ikinci derece çarpan Irreducible quadratic factor

İntegral tablosu Table of integrals

Bilgisayar cebiri sistemi Computer algebra system

Sayısal integrasyon Numerical integration

Yamuk kuralı Trapezoidal rule

Simpson kuralı Simpson's rule

Hata sınırı Error bound

Has olmayan integral Improper integral

Yakınsak integral Convergent integral

Iraksak integral Divergent integral

Doğrudan karşılaştırma testi Direct comparison test

Limit karşılaştırma testi Limit comparison test

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8. Ünite: Sonsuz Diziler ve Seriler

Dizi Sequence

Terim Term

Dizinin limiti Limit of a sequence

Yakınsak Convergent

Iraksak Divergent

Yakınsaklık Convergence

Iraksaklık Divergence

Monoton dizi Monotonic sequence

Sınırlı dizi Bounded sequence

Monoton dizi teoremi Monotonic sequence theorem

Özyinelemeli tanım Recursive definition

Sonsuz seri Infinite series

Kısmi toplam Partial sum

Geometrik seri Geometric series

Ortak oran Common ratio

Teleskopik seri Telescoping series

Harmonik seri Harmonic series

p-serisi p-series

Iraksaklık için n. terim testi nth-term test for divergence

İntegral testi Integral test

Karşılaştırma testi Comparison test

Oran testi Ratio test

Kök (Cauchy kök) testi Root test

Mutlak yakınsaklık Absolute convergence

Koşullu yakınsaklık Conditional convergence

Alterne (almaşık) seri Alternating series

Alterne (almaşık) seri testi Alternating series test

Kuvvet serisi Power series

Yakınsaklık yarıçapı Radius of convergence

Yakınsaklık aralığı Interval of convergence

Taylor serisi Taylor series

Maclaurin serisi Maclaurin series

Taylor polinomu Taylor polynomial

Kalan terim Remainder

Binom serisi Binomial series

Kalan tahmin teoremi Remainder estimation theorem

9. Ünite: Parametrik Denklemler ve Kutupsal Koordinatlar

Parametrik denklem Parametric equation

Parametre Parameter

Parametre aralığı Parameter interval

Parametrizasyon (parametrelendirme) Parametrization

Parametrik eğri Parametric curve

Başlangıç noktası Initial point

Bitiş noktası Terminal point

Sikloit Cycloid

Kutupsal koordinatlar Polar coordinates

Kutup Pole

Kutupsal eksen Polar axis

Kutupsal denklem Polar equation

Kardiyoit (kalp eğrisi) Cardioid

Gül eğrisi Rose curve

Limason Limaçon

Lemniskat Lemniscate

Simetri testi Symmetry test

Konik kesit Conic section

Parabol Parabola

Elips Ellipse

Hiperbol Hyperbola

Odak Focus

Doğrultman Directrix

Tepe noktası Vertex

Asimptot Asymptote

Dışmerkezlik Eccentricity

Büyük eksen Major axis

Küçük eksen Minor axis

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10. Ünite: Vektörler ve Uzay Geometrisi

Üç boyutlu koordinat sistemi Three-dimensional coordinate system

Sağ el kuralı Right-hand rule

Sekizde bir bölge Octant

Vektör Vector

Skaler Scalar

Bileşen Component

Birim vektör Unit vector

Standart birim vektörler Standard unit vectors

Uzunluk Length

Yön Direction

Nokta (iç) çarpımı Dot product

Skaler çarpım Scalar product

Vektörel (dış) çarpım Cross product

Vektör izdüşümü Vector projection

Skaler bileşen Scalar component

Dik (dikgen) vektörler Orthogonal vectors

Paralel vektörler Parallel vectors

Üçlü skaler çarpım Triple scalar product

Paralelkenar Parallelogram

Paralelyüzlü Parallelepiped

Doğru Line

Düzlem Plane

Normal vektör Normal vector

Aykırı doğrular Skew lines

Silindir Cylinder

Kuadrik (ikinci dereceden) yüzey Quadric surface

Elipsoit Ellipsoid

Paraboloit Paraboloid

Hiperboloit Hyperboloid

Hiperbolik paraboloit Hyperbolic paraboloid

Koni Cone

11. Ünite: Vektör Değerli Fonksiyonlar ve Uzayda Hareket

Vektör değerli fonksiyon Vector-valued function

Uzay eğrisi Space curve

Teğet vektör Tangent vector

Teğet doğru Tangent line

Hız vektörü Velocity vector

İvme vektörü Acceleration vector

Sürat Speed

Helis Helix

Eğik atış hareketi Projectile motion

Yay uzunluğu parametresi Arc length parameter

Birim teğet vektör Unit tangent vector

Eğrilik Curvature

Eğrilik yarıçapı Radius of curvature

Eğrilik merkezi Center of curvature

Eğrilik çemberi Circle of curvature

Birim normal vektör Principal unit normal vector

Binormal vektör (ikincil dikme) Binormal vector

Burulma Torsion

TNB çatısı TNB frame

Teğetsel bileşen Tangential component

Normal bileşen Normal component

Kepler yasaları Kepler's laws

12. Ünite: Kısmi Türevler

Çok değişkenli fonksiyon Function of several variables

Tanım kümesi Domain

Görüntü kümesi Range

Seviye (düzey) eğrisi Level curve

Seviye (düzey) yüzeyi Level surface

Kontur çizgisi Contour line

İç nokta Interior point

Sınır noktası Boundary point

Açık bölge Open region

Kapalı bölge Closed region

Sınırlı bölge Bounded region

İki yol testi Two-path test

Kısmi türev Partial derivative

İkinci mertebe kısmi türev Second-order partial derivative

Karışık türev Mixed derivative

Türevlenebilir Differentiable

Zincir kuralı Chain rule

Kapalı türev Implicit differentiation

Yönlü türev Directional derivative

Gradyan Gradient

Gradyan vektörü Gradient vector

Teğet düzlem Tangent plane

Normal doğru Normal line

Doğrusallaştırma Linearization

Toplam diferansiyel Total differential

Yerel maksimum Local maximum

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Yerel minimum Local minimum

Kritik nokta Critical point

Eyer noktası Saddle point

İkinci türev testi Second derivative test

Hessian determinantı Hessian

Mutlak ekstremum Absolute extremum

Lagrange çarpanları Lagrange multipliers

Kısıt Constraint

Kısıtlı değişken Constrained variable

13. Ünite: Katlı İntegraller

Katlı integral Multiple integral

Çift integral Double integral

Yinelenmiş integral Iterated integral

Fubini teoremi Fubini's theorem

Riemann toplamı Riemann sum

İntegrasyon bölgesi Region of integration

İntegrasyon sınırları Limits of integration

İntegrasyon sırası Order of integration

Ortalama değer Average value

Üçlü integral Triple integral

Kutupsal biçimde çift integral Double integral in polar form

Silindirik koordinatlar Cylindrical coordinates

Küresel koordinatlar Spherical coordinates

Hacim elemanı Volume element

Birinci moment First moment

Eylemsizlik momenti (dönme eylemsizliği) Moment of inertia

Değişken değiştirme Change of variables

Jakobiyen Jacobian

Dönüşüm Transformation

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14. Ünite: İntegraller ve Vektör Alanları

Eğrisel (hat) integral Line integral

Vektör alanı Vector field

Gradyan alanı Gradient field

İş Work

Dolanım (dolaşım) Circulation

Akı Flux

Yoldan bağımsızlık Path independence

Korunumlu alan Conservative field

Potansiyel fonksiyon Potential function

Tam diferansiyel Exact differential

Bağlantılı bölge Connected region

Basit bağlantılı bölge Simply connected region

Basit kapalı eğri Simple closed curve

Pozitif yönlendirme Positive orientation

Green teoremi Green's theorem

Rotasyonel (dolam) Curl

Diverjans (ıraksay) Divergence

Parametrize edilmiş yüzey Parametrized surface

Yüzey alanı elemanı Surface area element

Yüzey integrali Surface integral

Yönlendirilebilir yüzey Orientable surface

Stokes teoremi Stokes' theorem

Diverjans teoremi Divergence theorem