CHAPTER 5

APPLICATIONS OF INTEGRATION

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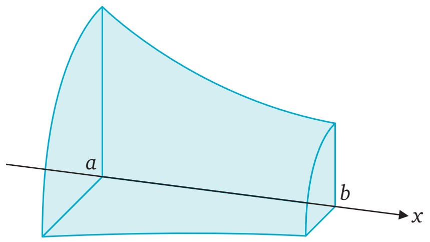
Volume Calculation by Cross-Sectional Area

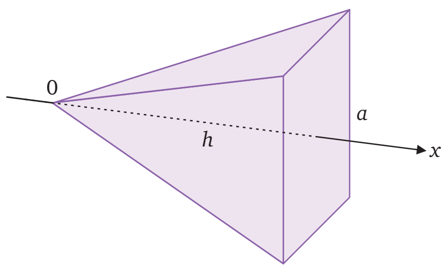
Let's discuss how to calculate volume by slicing with parallel surfaces. Let's define the mathematical definition of volume.



DEFINITION: The volume of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b:

$$V = \int_a^b A(x)\,dx$$

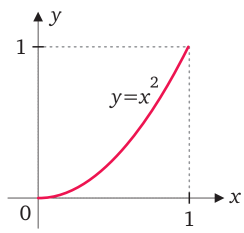
EXAMPLE 1: Calculate the volume of a square right pyramid with base edge length a and height h.



Solids of revolution: disk method

Let's examine what solids of revolution are and how they are obtained. Let's explain the disk method used in volume calculation and express it mathematically.

EXAMPLE 2: The region between the curve $y = x^2$, the interval $0 \le x \le 1$, and the x-axis is rotated around the x-axis to form a solid. Calculate the volume of this solid of revolution using an integral.



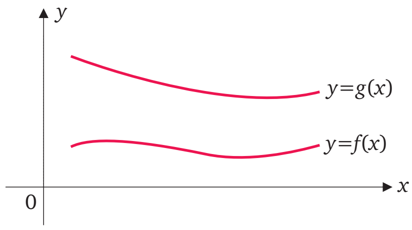
EXAMPLE 3: Consider the region bounded by the curve $y = \sqrt{x}$, the line $y = 2$, and the line $x = 0$. Write the integral that gives the volume of the solid of revolution formed by rotating this region around the line $y = 2$.

Let's mathematically express the volume calculation for the case of rotation around the y-axis.

EXAMPLE 4: Consider the region between the parabola $x = y^2$ and the line $x = 1$. Write the integral that gives the volume of the solid of revolution formed by rotating this region around the line $x = 1$.

Solids of revolution: washer method

Let us examine how to calculate the volumes of solids of revolution with hollow centers using the washer method.



EXAMPLE 5: Consider the region bounded by the curve $y = e^x$, the line $y = x + 1$, and the line $x = 1$. Write the integral that gives the volume of the solid of revolution formed by rotating this region around the x-axis.

EXAMPLE 6: Consider the region remaining between the parabola $x = y^2$ and the line $x = 1$. Write the integral that gives the volume of the solid of revolution formed by rotating this region around the line $x = 2$.

Volume Calculation with Cylindrical Shells

Let's obtain a solid of revolution by rotating the region between the parabola $y=4-x^2$ and the x-axis around the line $x=3$. Let's discuss how we can find the volume of this solid.

Let's express the cylindrical shell method mathematically.

EXAMPLE 7: Consider the region between the curve $y=\sqrt{x}$, the x-axis, and the line $x=1$. Write the integral that gives the volume of the solid of revolution formed by rotating this region around the y-axis using the cylindrical shell method.

EXAMPLE 8: Write the integral that gives the volume of the solid formed by rotating the region between the curve $x=y-y^2$ and the y-axis around the x-axis using the cylindrical shell method.

Arc Length

Let's discuss how the length of a curve is obtained. Then, let's define the length of a curve.

DEFINITION: If the derivative f' is continuous on the closed interval $[a,b]$, the length of the curve $y=f(x)$ from point $A=(a,f(a))$ to point $B=(b,f(b))$ is the value of the following integral:

$$L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

EXAMPLE 9: Write the integral that gives the length of the curve $y=x^2-x$ in the interval from $x=0$ to $x=2$.

EXAMPLE 10: Find the length of the curve $y=\frac{2}{3}x^{3/2}$ in the interval from $x=0$ to $x=3$.

Areas of Surfaces of Revolution

Let's discuss how we obtain the surface area of solids of revolution. Then, let's examine the mathematical definition of surface area for solids of revolution.

DEFINITION: If the function $f(x) \geq 0$ is continuously differentiable on the interval $[a, b]$, the surface area formed by rotating the graph of $y = f(x)$ around the x-axis is the value of the following integral:

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$$

EXAMPLE 11: Find the area of the surface formed by rotating the part of the curve $y = \sqrt{x}$ in the interval $1 \leq x \leq 2$ around the x-axis.

Physical Work and Pressure Force Calculations

Let's define the total work done by a variable force $F(x)$ while moving an object from point $x = a$ to point $x = b$.

EXAMPLE 12: Find the work required to compress a spring with a natural length of 40 cm and a force constant of $k = 300$ N/m to a length of 20 cm.

EXAMPLE 13: A rope 10 m long with a mass per unit length of 0.1 kg/m is hanging from the top of a building. How many joules of work are required to pull this entire rope to the top of the building? (Take the magnitude of the gravitational acceleration as $g = 10$ m/s^2.)

EXAMPLE 14: The water in a full rectangular prism-shaped pool with dimensions 8 m × 12 m and a depth of 2 m will be emptied by pumping it to the top edge level of the pool. Assuming the specific weight of water is 10 kN/m^3, how many joules of work are required for this process?

Let's examine how we can utilize integrals when calculating pressure force.

EXAMPLE 15: A flat isosceles triangular plate with a base of 3 m and a height of 2 m is placed vertically in water with its base at the top. The upper edge of the plate (the triangle base) is 1 m below the water surface. Assuming the specific weight of water is 10 kN/m^3, find the total force exerted by the water on one side of the plate.

Moments and Center of Mass

Let's discuss how to find the center of mass of a one-dimensional object and the concept of moments. Let's express the position of the center of mass mathematically.

EXAMPLE 16: For a rod located on the x-axis in the interval $[0,2]$ with a position-dependent mass density $\delta(x) = 4 + 2x^3$;

a. Find its total mass.

b. Find the coordinate of its center of mass.

Let's discuss how to find the center of mass of a two-dimensional object. Let's express the position of the center of mass mathematically.

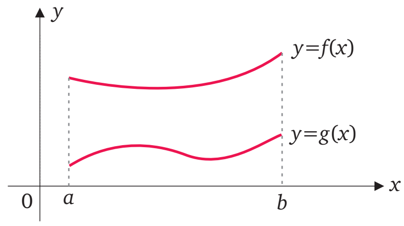
EXAMPLE 17: There is a triangular plate in the coordinate plane bounded by the x-axis, the line $x = 2$, and the line $y = 3x$. Given that the position-dependent density of this plate is $\delta(x) = x + 2$;

a. Find the total mass of the plate.

b. Find the x-coordinate of the center of mass of the plate.

c. Find the y-coordinate of the center of mass of the plate.

Let's discuss how to find the center of mass of plates remaining between two curves.



EXAMPLE 18: In the coordinate plane, there is a plate bounded from above by the curve $f(x) = \sqrt{x}$ and from below by the curve $g(x) = x^2$. Given that the position-dependent density function of this plate is $\delta(x) = 6x$;

a. Find the total mass of the plate.

b. Find the x-coordinate of the center of mass of the plate.

c. Find the y-coordinate of the center of mass of the plate.

Centroids

Let us discuss how to find the centroid of a shape.

EXAMPLE 19: There is a plate in the coordinate plane located above the x-axis, bounded by a semicircle of radius $r = 4$ centered at the origin and the x-axis. The density of this plate is equal to the constant δ. For this plate;

a. Find its total mass.

b. Find the coordinates of its center.

Exercises and Problems

1. In the first quadrant, consider the region bounded on the left by the circle $x^2 + y^2 = 4$, on the right by the line $x = 2$, and from above by the line $y = 2$. Write the integral that gives the volume of the solid of revolution formed by rotating this region about the y-axis;

a. Using the washer method,

b. Using the cylindrical shells method.

2. Consider the region bounded by the parabola $y = x^2 + 1$ and the line $y = -x + 3$. Write the integral that gives the volume of the solid of revolution formed by rotating this region about the x-axis.

3. To calculate the volume of the solid of revolution formed by rotating the triangular region with vertices $(1,0)$, $(2,1)$, and $(1,1)$ about the y-axis;

a. Write the integral giving the volume using the disk/washer method.

b. Write the integral giving the volume using the cylindrical shells method.

4. Find the length of the graph of the function $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ over the interval $1 \leq x \leq 3$.

5. Find the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ over the interval from $x = 0$ to $x = \ln(3)$.

6. Write the integral required to find the circumference of a circle with radius r centered at the origin.

7. Let S be the surface obtained by rotating the part of the curve $y = \frac{1}{3}x^3$ on the interval $0 \leq x \leq 1$ about the x-axis. Calculate the area of this surface.

8. Let region D be a region bounded by the curves $y = e^x$, $y = \sqrt{x+1}$ and the line $y = 2$.

a. Express the volume of the solid S formed by rotating region D about the y-axis using one or more integrals with the slicing (disk/washer) method.

b. Express the volume of the solid S given in part (a) using one or more integrals with the cylindrical shells method.

c. Express the volume of the solid formed by rotating region D about the line $x = -2$ using one or more integrals with the slicing (disk/washer) method.

9. The region remaining inside the circle given by the equation $x^2-6x+y^2=-5$ is rotated about the y-axis. Write the integral that gives the volume of the solid formed as a result of this rotation using the cylindrical shells method.

10. Let region $\mathscr{R}$ be the region bounded by the lines $x=0$ and $y=x-3$ and the curve $y=(x-3)^2(x+1)$.

a. Express the volume of the solid formed by rotating region $\mathscr{R}$ about the y-axis as a definite integral.

b. Express the volume of the solid formed by rotating region $\mathscr{R}$ about the line $y=-3$ as a definite integral.

Answers

Example questions

1	$V=\frac{1}{3}a^2h$
2	$V=\frac{\pi}{5}$
3	$V=\pi\int_0^4(2-\sqrt{x})^2dx$
4	$V=\pi\int_{-1}^1(1-y^2)^2\,dy$
5	$V=\pi\int_0^1\left[e^{2x}-(x+1)^2\right]dx$
6	$V=\pi\int_{-1}^1\left[(2-y^2)^2-1\right]dy$
7	$V=\int_0^1 2\pi x\sqrt{x}\,dx$
8	$V=2\pi\int_0^1(y^2-y^3)\,dy$
9	$L=\int_0^2\sqrt{1+(2x-1)^2}\,dx$
10	$L=\frac{14}{3}$
11	$S=\frac{\pi}{6}(27-5\sqrt{5})$
12	$W=6\text{ J}$
13	$W=50\text{ J}$
14	$W=1920\text{ kJ}$
15	$F=50\text{ kN}$
16	a. $M=16$ b. $\bar{x}=1.3$
17	a. $M=20$ b. $\bar{x}=1.4$ c. $\bar{y}=2.1$
18	a. $M=0.9$ b. $\bar{x}=\frac{4}{7}$ c. $\bar{y}=\frac{5}{9}$
19	a. $M=8\pi\delta$ b. $(0,\frac{16}{3\pi})$

Exercises and problems

1	a. $V=\pi\int_0^2 y^2\,dy$ b. $V=2\pi\int_0^2 x\left(2-\sqrt{4-x^2}\right)dx$
2	$V=\pi\int_{-2}^1\left[(-x+3)^2-(x^2+1)^2\right]dx$
3	a. $V=\pi\int_0^1\left[(y+1)^2-1\right]dy$ b. $V=2\pi\int_1^2 x(2-x)dx$
4	$L=\frac{14}{3}$

5	$L = \frac{4}{3}$
6	$C = 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx$
7	$A = \frac{pi}{9}(2\sqrt{2}-1)$
8	a. $V = \pi\int_1^2 \left[(y^2-1)^2-(\ln y)^2\right]dy$ b. $V = 2\pi\int_0^{\ln 2} x\left(e^x-\sqrt{x+1}\right)dx$ $+2\pi\int_{\ln 2}^3 x\left(2-\sqrt{x+1}\right)dx$ c. $V = \pi\int_1^2 \left[(y^2+1)^2-(\ln y+2)^2\right]dy$
9	$V = 4\pi\int_1^5 x\sqrt{4-(x-3)^2}\,dx$
10	a. $V = 2\pi\int_0^3 x\left[(x-3)^2(x+1)-(x-3)\right]dx$ b. $V = \pi\int_0^3 \left(\left[(x-3)^2(x+1)+3\right]^2-x^2\right)dx$

CHAPTER 7

TECHNIQUES OF INTEGRATION

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Using Basic Integration Formulas

Let's examine the basic integration formulas.

1. $\int k\,dx = kx + C$ (for any k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

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EXAMPLE 1: Evaluate the indefinite integrals given below using appropriate substitution methods and basic integral formulas:

a. $\int \frac{e^x}{9+e^{2x}}\,dx$ b. $\int \frac{1}{\sqrt{4-25x^2}}\,dx$

EXAMPLE 2: Evaluate the following integral using the method of completing the square:

$$\int \frac{dx}{\sqrt{3-2x-x^2}}$$

EXAMPLE 3: Evaluate the following indefinite integral:

$$\int \frac{1}{1+\cos x}dx$$

EXAMPLE 4: Evaluate the following indefinite integral:

$$\int \frac{4x^2-5x+3}{2x-1}\,dx$$

EXAMPLE 5: Evaluate the following indefinite integral:

$$\int \frac{dx}{(2+\sqrt{x})^2}$$

Integration by Parts

Let's derive the integration by parts formula. Let's write the differential form of the formula.

EXAMPLE 6: Evaluate the following indefinite integral:

$$\int x\sin x\,dx$$

EXAMPLE 7: Evaluate the following indefinite integral:

$$\int \ln x\,dx$$

EXAMPLE 8: Evaluate the following indefinite integral:

$$\int x^2 e^x \, dx$$

EXAMPLE 9: Evaluate the following indefinite integral:

$$\int e^x \sin x \, dx$$

EXAMPLE 10: Derive a formula that expresses the following integral in terms of an integral of a lower power of $\cos x$:

$$\int \cos^n x \, dx$$

Let's examine how to evaluate definite integrals using the integration by parts method.

EXAMPLE 11: Consider the region bounded by the curve $y = xe^{-x}$, the x-axis, and the vertical line $x = 5$;

a. Express its area as a definite integral.

b. Evaluate the integral using integration by parts.

Trigonometric Integrals

Let m and n be integers; let's examine the solution strategies for the following integral:

$$\int \sin^m x \cos^n x \, dx$$

EXAMPLE 12: Evaluate the following indefinite integral:

$$\int \cos^3 x \sin^2 x \, dx$$

EXAMPLE 13: Evaluate the following indefinite integral:

$$\int \cos^3 x \, dx$$

EXAMPLE 14: Evaluate the following indefinite integral:

$$\int \sin^4 x \cos^2 x \, dx$$

Let's examine how to eliminate the radical in trigonometric integrals containing radical expressions.

EXAMPLE 15: Evaluate the following indefinite integral.

$$\int \sqrt{1-\cos x} \, dx$$

Let's discuss how to evaluate integrals involving powers of $\tan x$ and $\sec x$ functions.

EXAMPLE 16: Evaluate the following indefinite integral.

$$\int \tan^5 x \, dx$$

EXAMPLE 17: Evaluate the following indefinite integral.

$$\int \sec^3 x\,dx$$

We can use product-to-sum formulas when evaluating the integrals $\int \sin mx \sin nx\,dx$, $\int \sin mx \cos nx\,dx$, and $\int \cos mx \cos nx\,dx$.

- $\sin x \cdot \cos y = \frac{1}{2}[\sin(x+y)+\sin(x-y)]$
- $\cos x \cdot \cos y = \frac{1}{2}[\cos(x+y)+\cos(x-y)]$
- $\sin x \cdot \sin y = \frac{1}{2}[\cos(x-y)-\cos(x+y)]$

EXAMPLE 18: Evaluate the following indefinite integral:

$$\int \sin x \cos 3x\,dx$$

Trigonometric Substitutions

Let's examine how to apply trigonometric substitutions through examples.

EXAMPLE 19: Evaluate the following indefinite integral:

$$\int \frac{1}{1+x^2}dx$$

EXAMPLE 20: Evaluate the following indefinite integral for $a > 0$:

$$\int \frac{dx}{\sqrt{a^2+x^2}}$$

EXAMPLE 21: Evaluate the following indefinite integral:

$$\int \frac{\sqrt{16-x^2}}{x^2}\,dx$$

Integration of Rational Functions by Partial Fractions

Let's examine how to apply the method of partial fraction decomposition through examples.

EXAMPLE 22: Evaluate the following indefinite integral:

$$\int \frac{7x+1}{(x-2)(x+3)}\,dx$$

EXAMPLE 23: Evaluate the following indefinite integral:

$$\int \frac{3x-5}{(x^2+1)(x+1)^2}\,dx$$

Improper Integrals

Let's discuss how integrals with infinite limits can yield finite results through an example.

EXAMPLE 24: Consider the region under the curve $y = \frac{1}{x^2}$ extending from the line $x = 1$ to infinity to the right ($x \geq 1$). To determine whether this region has a finite area;

a. Find the function $A(b)$ that gives the area of the part of the region between $x = 1$ and $x = b$ $(b > 1)$.

b. Find the total area by calculating the limit of this area as $b \to \infty$.

Let's define **improper integrals of Type 1**. Let's discuss what it means for these integrals to be **convergent** or **divergent**.

EXAMPLE 25: Is the area under the function $f(x) = xe^{-x}$ on the interval from $x = 1$ to $x = \infty$ finite? If so, find the value of this area.

$$\int_1^\infty \frac{dx}{x^p}$$

Determine for which values of p the integral is convergent or divergent.

Let's discuss how the presence of a vertical asymptote in the integrand can lead to improper integrals. Let's examine the following integral:

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

Let's define **improper integrals of Type 2**. Let's discuss what it means for these integrals to be convergent or divergent.

EXAMPLE 26: Determine whether the improper integral below converges or diverges:

$$\int_0^2 \frac{1}{2-x}\,dx$$

EXAMPLE 27: Determine whether the following improper integral converges or diverges:

$$I = \int_0^2 \frac{dx}{\sqrt{|x-1|}}$$

Convergence and divergence tests

Let's explain the **Direct Comparison Test**.

THEOREM: Let functions f and g be continuous on the interval $[a, \infty)$ and $0 \le f(x) \le g(x)$ for all $x \ge a$. In this case:

a. If $\int_a^\infty g(x)\,dx$ is convergent, then $\int_a^\infty f(x)\,dx$ is also convergent.

b. If $\int_a^\infty f(x)\,dx$ is divergent, then $\int_a^\infty g(x)\,dx$ is also divergent.

EXAMPLE 28: Use the Direct Comparison Test to determine whether each improper integral converges or diverges.

a. $\int_1^\infty \frac{1}{x^3+1}\,dx$

c. $\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx$

b. $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

d. $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$

Let's explain the **Limit Comparison Test**.

THEOREM: Let f and g be positive continuous functions on the interval $[a, \infty)$. If,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

is satisfied;

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both integrals either converge or both diverge.

EXAMPLE 29: Use the Limit Comparison Test to determine whether each improper integral converges or diverges.

a. $\int_2^\infty \frac{1}{x^2-1}\,dx$ b. $\int_1^\infty \frac{x+2}{2x^2+1}\,dx$

Exercises and Problems

1. Evaluate the following indefinite integrals.

a. $\int \frac{dx}{\sqrt{1-4x^2}}$ c. $\int \frac{e^{-\tan x}}{\cos^2 x}\,dx$

b. $\int (\sec x + \tan x)^2 dx$ d. $\int \frac{dx}{e^{2x}+e^{-2x}}$

2. Evaluate the following integrals using the method of integration by parts.

a. $\int x^2 \ln(x)\,dx$ c. $\int e^{3x}\cos x\,dx$

b. $\int (2x-3)\sin 2x\,dx$ d. $\int x^2 e^{-x}\,dx$

3. Let n be a positive integer; show that the following equality holds using the method of integration by parts:

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

4. Let f be a continuous function; show that the following equality holds using the method of integration by parts:

$$\int_a^b \left(\int_x^b f(t)\,dt\right)dx = \int_a^b (x-a)f(x)\,dx$$

5. Evaluate the following trigonometric integrals.

a. $\int \sin^5 x\,dx$

b. $\int \frac{\sin^2 x}{\sqrt{1-\cos x}}\,dx$

c. $\int \tan^4 x \sec^4 x\,dx$

d. $\int x\sin^3 x\,dx$

6. Evaluate the following integrals using trigonometric substitutions.

a. $\int \frac{dx}{x\sqrt{x^2-1}}$

b. $\int \sqrt{\frac{x+1}{1-x}}\,dx$

c. $\int \frac{dx}{(9-x^2)^{3/2}}$

d. $\int \frac{2dx}{(4x^2+1)^2}$

7. Evaluate the following integrals using the method of partial fraction decomposition.

a. $\int \frac{1}{x(x^2+1)}\,dx$

b. $\int \frac{x+7}{x^2+2x-8}\,dx$

c. $\int \frac{x^4}{x^2-4}\,dx$

8. Determine whether each of the following improper integrals is convergent or divergent using the direct comparison test.

a. $\int_1^\infty \frac{2+\sin\left(x^2\right)}{\sqrt{x^5+4x}}\,dx$

b. $\int_1^\infty \frac{e^{1/x}}{x+\ln^2 x}\,dx$

c. $\int_0^1 \frac{1}{\sqrt{x}\ln(1+x)}\,dx$

d. $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$

9. Find the values of p for which the following integral converges and evaluate the integral for these values.

$$\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$$

10. Use the Limit Comparison Test to determine whether the following improper integral converges or diverges:

$$\int_2^\infty \frac{\ln(x)+\sqrt{x}}{x^2-\ln(x)}\,dx$$

Answers

Example questions

1	a. $\frac{1}{3}\arctan\left(\frac{e^x}{3}\right)+C$ b. $\frac{1}{5}\arcsin\left(\frac{5x}{2}\right)+C$
2	$\arcsin\left(\frac{x+1}{2}\right)+C$
3	$\csc x-\cot x+C$
4	$x^2-\frac{3}{2}x+\frac{3}{4}\ln\|2x-1\|+C$
5	$2\ln\left(2+\sqrt{x}\right)+\frac{4}{2+\sqrt{x}}+C$
6	$-x\cos x+\sin x+C$
7	$x\ln x-x+C$
8	$(x^2-2x+2)e^x+C$
9	$\frac{e^x(\sin x-\cos x)}{2}+C$
10	$\int\cos^n x\,dx=\frac{\cos^{n-1}x\sin x}{n}+\frac{n-1}{n}\int\cos^{n-2}x\,dx$
11	a. $A=\int_0^5 xe^{-x}\,dx$ b. $A=1-6e^{-5}$
12	$\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$
13	$\sin x-\frac{1}{3}\sin^3 x+C$
14	$\frac{x}{16}-\frac{\sin(4x)}{64}-\frac{\sin^3(2x)}{48}+C$
15	$-2\sqrt{2}\cos\left(\frac{x}{2}\right)+C$
16	$\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\ln\|\sec x\|+C$
17	$\frac{1}{2}(\sec x\tan x+\ln\|\sec x+\tan x\|)+C$
18	$-\frac{1}{8}\cos 4x+\frac{1}{4}\cos 2x+C$
19	$\arctan x+C$
20	$\ln\left(x+\sqrt{a^2+x^2}\right)+C$
21	$-\frac{\sqrt{16-x^2}}{x}-\arcsin\left(\frac{x}{4}\right)+C$
22	$3\ln\|x-2\|+4\ln\|x+3\|+C$
23	$\frac{5}{4}\ln\left(x^2+1\right)+\frac{3}{2}\arctan(x)-\frac{5}{2}\ln\|x+1\|+\frac{4}{x+1}+C$
24	a. $A(b)=1-\frac{1}{b}$ b. Area $=1$
25	It is finite, Area $=\frac{2}{e}$
26	The integral is divergent.
27	The integral is convergent, $I=4$
28	a. Convergent b. Convergent c. Divergent d. Convergent
29	a. Convergent b. Divergent

Exercises and problems

1	a. $\frac{1}{2}\arcsin(2x)+C$ b. $2\tan x+2\sec x-x+C$ c. $-e^{-\tan x}+C$ d. $\frac{1}{2}\arctan\left(e^{2x}\right)+C$
2	a. $\frac{x^3}{3}\ln(x)-\frac{x^3}{9}+C$ b. $-\frac{2x-3}{2}\cos 2x+\frac{1}{2}\sin 2x+C$ c. $\frac{e^{3x}}{10}(\sin x+3\cos x)+C$ d. $-e^{-x}(x^2+2x+2)+C$
3	Refer to the video.
4	Refer to the video.
5	a. $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+C$ b. $-\frac{4\sqrt{2}}{3}\cos^3\left(\frac{x}{2}\right)+C$ c. $\frac{1}{5}\tan^5 x+\frac{1}{7}\tan^7 x+C$ d. $-x\cos x+\frac{x\cos^3 x}{3}+\frac{2}{3}\sin x+\frac{1}{9}\sin^3 x+C$
6	a. $\operatorname{arcsec}(x)+C$ b. $\arcsin(x)-\sqrt{1-x^2}+C$ c. $\frac{x}{9\sqrt{9-x^2}}+C$ d. $\frac{1}{2}\arctan(2x)+\frac{x}{4x^2+1}+C$
7	a. $\ln\|x\|-\frac{1}{2}\ln\left(x^2+1\right)+C$ b. $-\frac{1}{2}\ln\|x+4\|+\frac{3}{2}\ln\|x-2\|+C$ c. $\frac{x^3}{3}+4x+4\ln\left\|\frac{x-2}{x+2}\right\|+C$
8	a. Convergent b. Divergent c. Divergent d. Divergent
9	Converges for $p>1$. Integral value: $\frac{1}{p-1}$
10	The integral is convergent, refer to the video.

CHAPTER 8

INFINITE SEQUENCES & SERIES - 1

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Motivation

Our main goal in this unit is:

- To learn how to represent complex functions using infinite sums (series).
- To measure how close an approximation (estimation) is to the actual value using "error estimation".
- To make differential equations and integral calculus much easier to solve by using series.

Let's examine the following examples:

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Sequences

Let's express what the concept of a sequence is.

Let's examine the following sequence examples.

- $a_n = 3n + 1$
- $b_n = (-1)^n \cdot n^2$

EXAMPLE 1: Find a formula for the general term a_n of the sequence whose first few terms are given below (assume the pattern continues in the same way):

$$\left\{ \frac{5}{3}, -\frac{7}{9}, \frac{9}{27}, -\frac{11}{81}, \frac{13}{243}, \cdots \right\}$$

Convergence and divergence

DEFINITION: A sequence $\{a_n\}$ is said to converge to the number L if for every number $\epsilon > 0$ there exists an integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

This is denoted by $\lim_{n\to\infty} a_n = L$ or $a_n \to L$, and the number L is called the limit of the sequence. If no such L exists, the sequence $\{a_n\}$ diverges.

Let's show that the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\}$ given by the general term $a_n = \frac{1}{n}$ is convergent.

Let's show that the sequence $\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ given by the general term $a_n = (-1)^{n+1}$ is divergent.

Let's define divergence to infinity or negative infinity for a sequence.

Calculating limits of sequences

Let's examine how we benefit from the methods used in limits when calculating the limits of sequences.

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, where A and B are real numbers, the following rules apply:

Sum Rule: $\lim_{n\to\infty}(a_n + b_n) = A + B$

Difference Rule: $\lim_{n\to\infty}(a_n - b_n) = A - B$

Constant Multiple Rule: $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$, (for any number k)

Product Rule: $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$

Quotient Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, for $B \neq 0$

NOTE: We can use many of the methods used in limit calculation here.

THEOREM: If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $a_n = f(n)$ defined for every integer n is also $\lim_{n\to\infty} a_n = L$.

EXAMPLE 2: Calculate the limits given below.

a. $\lim_{n\to\infty}\left(\frac{3}{n} - 2\right)$

b. $\lim_{n\to\infty} \frac{5n^4-2}{3n^4+n^2}$

c. $\lim_{n\to\infty} \frac{\sin n}{n}$

d. $\lim_{n\to\infty}(-1)^n \frac{1}{n}$

Let's calculate the following limit using L'Hôpital's rule. Let's discuss the condition for using this rule.

$$\lim_{n\to\infty} \frac{\ln n}{n}$$

Recursive definitions

Let's examine recursive definitions through an example.

EXAMPLE 3: The Fibonacci sequence $\{f_n\}$ is given, defined by $f_1 = 1$ and $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for every $n \geq 3$. Write the first 6 terms of the sequence accordingly.

Bounded monotonic sequences

Let's examine what it means for a sequence to be unbounded or bounded from above or below through examples.

- $a_n = n^2$
- $b_n = \frac{n+1}{n}$

Let's define what it means for a sequence to be monotonic or non-decreasing or non-increasing. Let's evaluate the following examples from this perspective.

- $\{n^3\} = 1, 8, 27, \dots, n^3, \dots$
- $\{\frac{n+1}{n}\} = 2, \frac{3}{2}, \frac{4}{3}, \dots, \frac{n+1}{n}, \dots$
- $\{\frac{1}{3^n}\} = \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots, \frac{1}{3^n}, \dots$
- $\{5\} = 5, 5, 5, \dots, 5, \dots$
- $\{(-2)^n\} = -2, 4, -8, 16, \dots$

THEOREM: If a sequence $\{a_n\}$ is both bounded and monotonic, then it is convergent.

Infinite Series

Let's explain the concept of an infinite series. Let's examine an example of an infinite series through partial sums.

Let's define convergence and divergence for a series.

Geometric series

Let's examine the form of geometric series. Let's write an example appropriate for this form. Let's discuss the convergence condition of such a series. For the case of convergence, let's obtain the value to which the series converges.

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

EXAMPLE 4: Find the sums of the infinite series whose general terms are given below.

a. $\sum_{n=1}^{\infty} 4\left(\frac{1}{3}\right)^{n-1}$ b. $\sum_{n=0}^{\infty} (-1)^n \frac{3}{4^n}$

EXAMPLE 5: A ball released from a height h above the ground rebounds to a height k times the height of the previous fall each time it hits the ground $(0 < k < 1)$. Find the total distance traveled by the ball in terms of h and k.

EXAMPLE 6: Express the repeating decimal 2.3575757... as the sum of an infinite geometric series and write it as a ratio of two integers.

EXAMPLE 7: Find the sum of the "telescoping" series given below:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$$

Divergence test (nth term test)

Let's examine why the following series is divergent.

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots$$

THEOREM: If $\sum_{n=1}^{\infty} a_n$ is convergent, then $a_n \to 0$.

EXAMPLE 8: Determine the convergence or divergence status of the infinite series given below.

a. $\sum_{n=1}^{\infty} (-1)^{n+1}$ b. $\sum_{n=1}^{\infty} \frac{-n}{2n+3}$

Combining series

When we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to obtain new convergent series.

THEOREM: If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then the following rules apply:

Sum Rule: $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$

Difference Rule: $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$

Constant Multiple Rule: $\sum k a_n = k \sum a_n = kA$ $(k \in \mathbb{R})$.

We can discuss this theorem for divergent series.

EXAMPLE 9: Find the sum of the following series:

$$\sum_{n=1}^{\infty} \frac{2^{n+1}+3}{5^{n-1}}$$

Adding or deleting terms and reindexing

Let us examine how we perform the operations of adding or removing terms and changing indices.

EXAMPLE 10: Examine the convergence of the infinite series given below and find the sum of the series if it converges.

a. $\sum_{n=2}^{\infty} \frac{1}{4^n}$

b. $\sum_{n=2}^{\infty} \frac{2^{n+1}}{3^n}$

Integral Test

Let us use the following theorem to show the divergence of the harmonic series.

THEOREM: Let $\sum_{n=1}^{\infty} a_n$ be a series consisting of non-negative terms. This series is convergent if and only if its partial sums are bounded from above.

EXAMPLE 11: Let us examine the convergence of the series given below:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

THEOREM: Let $\{a_n\}$ be a sequence with positive terms. Suppose that for all $x \geq N$ $(N \in \mathbb{Z}^+)$, the function f is a continuous, positive, and decreasing function and satisfies $a_n = f(n)$. In this case, the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ either both converge or both diverge.

Let us examine the convergence and divergence condition of the p-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

EXAMPLE 12: Determine the convergence or divergence of the infinite series given below.

a. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

b. $\sum_{n=2}^{\infty} \frac{1}{n \ln^2(n)}$

Error estimation with the integral test

Suppose that a series $\sum a_n$ with positive terms is shown to be convergent by the integral test. In this case, let us estimate the magnitude of the remainder term R_n, which measures the difference between the sum of the series S and the n-th partial sum s_n.

EXAMPLE 13: The sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ will be estimated using the error estimation method with the integral test by taking $n = 4$. Determine the lower and upper bounds for the sum of the series S. (Compare the result you obtained with the value $\frac{\pi^2}{6} \approx 1.64493$.)

Comparison Tests

Direct comparison test

THEOREM: Let $\sum a_n$ and $\sum b_n$ be two series satisfying the condition $0 \le a_n \le b_n$ for every n. In this case:

I. If $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.

II. If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

EXAMPLE 14: Examine the convergence of the infinite series given below using the direct comparison test.

a. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ b. $\sum_{n=1}^{\infty} \frac{1}{2^n+\sqrt{n}}$

Limit comparison test

THEOREM: Let $a_n > 0$ and $b_n > 0$ for every $n \geq N$ $(N \in \mathbb{Z})$.

I. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

II. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

III. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

EXAMPLE 15: Examine the convergence of the infinite series given below using the limit comparison test.

a. $\sum_{n=1}^{\infty} \frac{3n^2+5n}{n^4+2n-1}$ c. $\sum_{n=1}^{\infty} \frac{n+1}{n\ln(n)}$

b. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$

Absolute Convergence; The Ratio and Root Tests

DEFINITION: If the series $\sum |a_n|$ converges, then $\sum a_n$ converges absolutely.

THEOREM: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

EXAMPLE 16: Determine whether the series given below are absolutely convergent and investigate their convergence status.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$ b. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Ratio test

THEOREM: $\sum a_n$ be any series and suppose that:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$$

In this case;

I. If $\rho < 1$, the series is absolutely convergent.

II. If $\rho > 1$ or ρ is infinite, the series is divergent.

III. If $\rho = 1$, the test is inconclusive.

EXAMPLE 17: Investigate the convergence status of the series given below using the ratio test.

a. $\sum_{n=1}^{\infty} \frac{5^n}{n!}$ c. $\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$

b. $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

Root test

THEOREM: Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \rho$$

. In this case;

I. If $\rho < 1$, the series is absolutely convergent.

II. If $\rho > 1$ or ρ is infinite, the series is divergent.

III. If $\rho = 1$, the test is inconclusive.

EXAMPLE 18: Investigate the convergence status of the series given below using the root test.

a. $\sum_{n=1}^{\infty} \left(\frac{3n+1}{4n-2}\right)^n$ b. $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$

Alternating Series and Conditional Convergence

THEOREM:

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

A series of this form converges if the following conditions are met:

I. All u_n values must be positive.

II. The sequence u_n must satisfy the condition $u_n \geq u_{n+1}$ for every integer $n \geq N$ sufficiently large, i.e., it must exhibit a monotonically decreasing character.

III. $u_n \to 0$ must hold.

EXAMPLE 19: Investigate the convergence of the infinite series given below:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Let's examine the *alternating series estimation theorem* for the following infinite series, which is a variation of the harmonic series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Conditional convergence

DEFINITION: A series that is convergent but not absolutely convergent is called a conditionally convergent series.

Let's examine the following series for conditional convergence:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

EXAMPLE 20: Investigate the convergence status of the series given below.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$ c. $\sum_{n=1}^{\infty}(-1)^n \frac{\arctan n}{n^2+1}$

b. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n}{n+5}$

Exercises and Problems

1. Investigate the convergence status of the sequences whose general terms are given below. If convergent, find the limit.

a. $a_n = \frac{3n^2-5n+1}{2n^2+7}$

b. $a_n = \left(1+\frac{5}{n}\right)^n$

c. $a_n = \frac{\ln\left(n^2\right)}{n}$

d. $a_n = \frac{(-1)^n \sin(n)}{n^2+1}$

e. $a_n = \frac{n!}{n^n}$

f. $a_n = \frac{n^2+\ln n}{3n+2}$

2. Determine the monotonicity and boundedness of the sequences whose general terms are given below.

a. $a_n = \frac{4n-3}{2n+5}$

b. $a_n = \frac{(3n+1)!}{(n+2)!}$

3. Investigate the convergence status of the infinite series given below and find the sum of the series if it converges.

a. $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^{n+1}}{5^n}\right)$

b. $\sum_{n=2}^{\infty}\left(1-\frac{3}{2^n}\right)$

c. $\sum_{n=0}^{\infty}\frac{\cos n\pi}{2^n}$

d. $\sum_{n=1}^{\infty}\frac{5}{n(n+1)}$

e. $\sum_{n=1}^{\infty}\left(\frac{1}{2^{1/n}}-\frac{1}{2^{1/(n+1)}}\right)$

f. $\sum_{n=1}^{\infty}\ln\frac{1}{2^n}$

4. Apply the divergence test to the infinite series given below and state the result obtained.

a. $\sum_{n=0}^{\infty}\frac{e^n}{e^n+n^2}$

b. $\sum_{n=1}^{\infty}\frac{n(n+1)}{n^3}$

5. For each of the following geometric series, find the values of x for which the series converges. Also, express the sum of the series as a function of x for these x values.

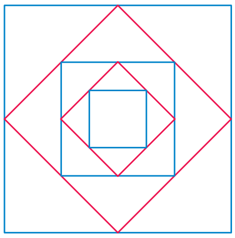
a. $\sum_{n=0}^{\infty}3^n x^n$

b. $\sum_{n=0}^{\infty}\left(\frac{x+2}{2}\right)^n$

c. $\sum_{n=0}^{\infty}\cos^n x$

d. $\sum_{n=0}^{\infty}\left(\frac{\ln x}{2}\right)^n$

6. Starting with a square of side length a, each new square is formed by joining the midpoints of the previous square. Find the total sum of the perimeters of all squares formed when this process is repeated infinitely, in terms of a.



7. Determine the convergence or divergence status of the infinite series given below using the integral test.

a. $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ b. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

8. Determine the convergence status of the series given below using comparison tests:

a. $\sum_{n=1}^{\infty} \frac{1}{2n^2+n}$ c. $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n} e^n}$

b. $\sum_{n=2}^{\infty} \frac{n+5}{n^2-n}$ d. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

9. Investigate the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n!}$. (Hint: First show that $1/n! \leq 1/n(n-1)$ for $n \geq 2$.)

10. Investigate the convergence status of the series given below using the ratio or root test.

a. $\sum_{n=1}^{\infty} \frac{(2n)!}{3^n (n!)^2}$ b. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

Answers

Example questions

1	$a_n = (-1)^{n+1} \frac{2n+3}{3^n}$
2	a. -2 b. $\frac{5}{3}$ c. 0 d. 0
3	$\{1,1,2,3,5,8\}$
4	a. 6 b. $\frac{12}{5}$
5	$S = h\left(\frac{1+k}{1-k}\right)$
6	$\frac{389}{165}$
7	$\frac{1}{2}$
8	a. Divergent b. Divergent
9	$\frac{125}{12}$
10	a. Convergent, $\frac{1}{12}$ b. Convergent, $\frac{8}{3}$
11	Convergent
12	a. Convergent b. Convergent
13	$1.62361 \leq S \leq 1.67361$
14	a. Divergent b. Convergent
15	a. Convergent b. Convergent c. Divergent
16	a. The series is absolutely convergent; therefore it is convergent. b. The series is absolutely convergent; therefore it is convergent.
17	a. Convergent b. Divergent c. Ratio test is inconclusive ($L = 1$)
18	a. Convergent b. Convergent
19	Convergent
20	a. Conditionally convergent b. Divergent c. Absolutely conver

Exercises and problems

1	a. Convergent, $L = \frac{3}{2}$ b. Convergent, $L = e^5$ c. Convergent, $L = 0$ d. Convergent, $L = 0$ e. Convergent, $L = 0$ f. Divergent, $+\infty$
2	a. Monotonically non-decreasing and bounded. Lower bound $\frac{1}{7}$, up b. Monotonically non-decreasing, bounded below by 4, unbounded
3	a. Convergent, $\frac{7}{6}$ b. Divergent c. Convergent, $\frac{2}{3}$ d. Convergent, 5 e. Convergent, $-\frac{1}{2}$ f. Divergent
4	a. Divergent b. $\lim_{n\to\infty} a_n = 0$, test is inconclusive
5	a. $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$, $S(x) = \frac{1}{1-3x}$ b. $x \in (-4, 0)$, $S(x) = -\frac{2}{x}$ c. $x \neq k\pi$ $(k \in \mathbb{Z})$, $S(x) = \frac{1}{1-\cos x}$ d. $x \in (e^{-2}, e^2)$, $S(x) = \frac{2}{2-\ln x}$
6	$L = 4a(2+\sqrt{2})$
7	a. Divergent b. Divergent
8	a. Convergent b. Divergent c. Convergent d. Divergent
9	Convergent
10	a. Divergent b. Convergent

CHAPTER 8

INFINITE SEQUENCES & SERIES - 2

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Power Series

DEFINITION: $x = 0$ centered power series is a series of the following form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

$x = a$ centered power series is a series of the following form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Here, a is the center and $c_0, c_1, c_2, \cdots, c_n, \cdots$ are constant coefficients.

Let us examine the following geometric power series and obtain the interval on which the series converges. For this interval, let us find the function to which the series converges.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

Let us examine the following geometric power series and obtain the interval on which the series converges. For this interval, let us find the function to which the series converges.

$$1 - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-1)^n + \cdots$$

EXAMPLE 1: Determine the interval of convergence for the power series given below.

a. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

b. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

c. $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

THEOREM: The convergence behavior of a power series $\sum_{n=0}^{\infty} a_n x^n$ is defined in terms of the behavior of the series at specific points as follows:

I. **Convergence:** If the series converges at $x = c$ $(c \neq 0)$, then for all values of x closer to the center than the point c, that is, for every $|x| < |c|$, the series converges absolutely.

II. **Divergence:** If the series diverges at $x = d$, then for all values of x farther from the center than the point d, that is, for every $|x| > |d|$, the series diverges.

NOTE: This theorem tells us that power series always have a convergence interval symmetric about the center.

Let us discuss the concept of the *radius of convergence* through the power series $\sum a_n(x-a)^n$.

Operations on Power Series

Let us examine the *series multiplication* operation for power series through the following example:

$$\left(\sum_{n=0}^{\infty}(-x)^n\right)\cdot\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\right)$$

In a power series, a function $f(x)$ can be substituted in place of the variable x.

Let us examine this on the following series for $f(x) = 4x^2$, together with its results:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

A power series can be differentiated term by term.

Let us examine this on the following series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

A power series can be integrated term by term.

Let us examine this on the following series:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots$$

EXAMPLE 2: Determine the function $f(x)$ defined by the following power series:

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots,\quad -1\le x\le 1.$$

EXAMPLE 3: The following series is given for the function $\sin x$ and converges to $\sin x$ for all values of x:

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\cdots$$

a. Find the first few terms of the series for $\cos x$. For which values of x does the series converge?

b. By substituting $2x$ for x in the series for $\sin x$, find a series that converges to $\sin 2x$ for all values of x.

c. Using series multiplication, compute the first terms of the series for $2\sin x\cos x$. Compare your answer with the answer in part (b).

Taylor and Maclaurin Series

Suppose that a function $f(x)$ is the sum of a power series with a positive radius of convergence around $x=a$. That is, let us write the function in the form

$$f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n=a_0+a_1(x-a)+a_2(x-a)^2+\cdots$$

Now let us obtain the coefficients here in terms of the derivatives of the function. Using these coefficients, let us define the Taylor and Maclaurin series.

EXAMPLE 4: Find the Taylor series of the function $f(x) = \ln x$ about the point $a = 1$ and determine the radius of convergence of this series.

f has derivatives of all orders at the point $x = a$. In this case, the Taylor polynomial of degree n, denoted by $P_n(x)$, representing the function f near the point a, is defined as follows:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Let us find the Taylor series generated at the point $x = 0$ for the function $f(x) = e^x$. Then, using the first three terms of this series, let us write the Taylor polynomials $P_0(x)$, $P_1(x)$, and $P_2(x)$.

EXAMPLE 5: Find the Maclaurin series expansion of the function $f(x) = \sin x$.

EXAMPLE 6: Find the Taylor expansion of the function $f(x) = x^3 - 5x + 1$ at the point $x = 1$.

Error estimation

Let us examine error estimates for Taylor series through examples.

EXAMPLE 7: Estimate the error that occurs when the polynomial $P_2(x) = 1 - \frac{x^2}{2}$ is used to approximate $\cos x$ at $x = 0.2$.

EXAMPLE 8: Approximate the value of $1/e$ so that the error is less than $1/100$.

THEOREM: If the function f and its first n derivatives $f', f'', \cdots, f^{(n)}$ are continuous on the closed interval between a and b, and if $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that the following equality holds:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

Let us show that the Taylor series of the function $\cos x$ at the point $x = 0$ converges to $\cos x$ for every value of x.

EXAMPLE 9: When values of x are small, the approximation $e^x \approx 1 + x + \frac{x^2}{2}$ is used. Estimate the error of this approximation when $|x| < 0.1$.

Applications of Taylor Series

Binomial series

$(1+x)^m$ has the following infinite series expansion for $|x|<1$:

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots$$

EXAMPLE 10: The function $f(x) = \sqrt{1+x}$ is given.

a. Write the first three terms of the Binomial series expansion of this function centered at $x = 0$.

b. Similarly, write the first three terms of the Binomial series expansion of the function $g(x) = \sqrt{1-x^2}$ centered at $x = 0$.

Evaluation of non-elementary integrals

Let us examine through an example how Taylor series are used in evaluating integrals.

EXAMPLE 11: Using the Maclaurin series of the function $\cos x^3$, answer the following:

a. Express the indefinite integral $\int \cos x^3\, dx$ as a power series.

b. Estimate the value of the definite integral $\int_0^1 \cos x^3\, dx$ with an error less than 10^{-2}.

Arctangents

Let us examine through an example how Taylor series are used in arctangent calculations.

EXAMPLE 12: Using the Taylor series of the function $f(x) = \arctan x$ about $x = 0$, suppose that we want to compute the value $\frac{\pi}{4}$ with an error smaller than 10^{-2}.

a. Write the Taylor series expansion of the function $\arctan x$.

b. Determine how many terms of the series must be summed to compute $\frac{\pi}{4}$ with the desired accuracy.

Evaluation of indeterminate forms

Sometimes we can evaluate indeterminate forms by expressing the relevant functions in terms of Taylor series. Let us examine the examples.

EXAMPLE 13: Evaluate the following limit using the Taylor series:

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

EXAMPLE 14: Evaluate the following limit using the Taylor series:

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

Euler's identity

Let us write Euler's identity and show this identity using Taylor series.

Important Maclaurin series and their radii of convergence:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \quad R = 1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots, \quad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots, \quad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots, \quad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \quad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \quad R = \infty$$

Exercises and Problems

1. For the power series given below:

i. Find the radius of convergence and the interval of convergence.

ii. For which values of x does the series converge absolutely?

iii. For which values of x does the series converge conditionally?

a. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 4^n}$

b. $\sum_{n=0}^{\infty} \frac{(-1)^n(2x+1)^n}{\sqrt{n+1}}$

c. $\sum_{n=1}^{\infty} \frac{x^n}{n^2 5^n}$

2. Find the interval of convergence of the series given below and express its sum as a function of x on this interval:

$$\sum_{n=0}^{\infty}(e^x - 1)^n$$

3. Find the sum of the series given below:

$$\frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \cdots + (-1)^{n-1}\frac{1}{n \cdot 2^n} + \cdots$$

4. Find the Taylor series (Maclaurin series) of the following functions at the point $x = 0$:

a. $f(x) = \frac{1}{1+x^3}$

b. $g(x) = e^{-x^2}$

5. Let the function $f(x) = \frac{3x^2}{3-x}$ be given.

a. Find the Maclaurin series (Taylor series about 0) of the function $f(x)$.

b. Find the interval of convergence of this series.

6. Let $G(x) = \int_0^x \sin t^2 \, dt$ be given.

a. Find the Maclaurin series (Taylor series about 0) of the function $G(x)$. (You may use the Maclaurin series of known functions.)

b. Approximate the value of $G(1)$ so that the error is less than 0.001.

Answers

Example questions

1	a. $[-1,1]$ b. $(-\infty,\infty)$ c. $\{0\}$
2	$f(x)=\arctan x$
3	Go to the video.
4	a. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x-1)^n$ b. $R=1$
5	$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$
6	$f(x)=-3-2(x-1)+3(x-1)^2+(x-1)^3$
7	Error $\leq\frac{(0.2)^4}{24}$
8	$\frac{3}{8}$
9	Error $\leq\frac{e^{0.1}}{6000}$
10	a. $1+\frac{1}{2}x-\frac{1}{8}x^2$ b. $1-\frac{1}{2}x^2-\frac{1}{8}x^4$
11	a. $C+x-\frac{x^7}{7\cdot 2!}+\frac{x^{13}}{13\cdot 4!}-\cdots$ b. ≈ 0.929
12	a. $\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ b. $n=50$ terms (last term $\frac{1}{99}$)
13	$\frac{1}{2}$
14	$\frac{1}{6}$

Exercises and problems

1	a. i. $R=4$, $[-1,7)$ ii. $(-1,7)$ iii. $x=-1$ b. i. $R=1/2$, $(-1,0]$ ii. $(-1,0)$ iii. $x=0$ c. i. $R=5$, $[-5,5]$ ii. $[-5,5]$ iii. $\emptyset$
2	$(-\infty,\ln 2)$, $f(x)=\frac{1}{2-e^x}$
3	Sum $=\ln\left(\frac{3}{2}\right)$
4	a. $\sum_{n=0}^{\infty}(-1)^n x^{3n}$ b. $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$
5	a. $\sum_{n=0}^{\infty}\frac{x^{n+2}}{3^n}$ b. $(-3,3)$
6	a. $G(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$ b. $G(1)\approx\frac{13}{42}\approx 0.3095$

CHAPTER 9

PARAMETRIC EQUATIONS AND POLAR COORDINATES

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Parametrization of Plane Curves

DEFINITION: As the variable t takes values in a certain interval, the set of points (x, y) determined by the equations $x = f(t)$ and $y = g(t)$ is called a parametric curve. These equations are called the parametric equations of the curve.

EXAMPLE 1: Considering the parabolic curve given by the equation $y = x^2$, answer the following questions:

a. Using the parameter $x = t$, write the parametric equations of the curve for $t \in (-\infty, \infty)$.

b. Using the parametrization $x = \sqrt{t}$, write the parametric equations of the curve for $t \geq 0$ and state which part of the curve this parametrization represents.

EXAMPLE 2: Sketch the curve defined by the parametric equations given below:

$$x = t^2 - 1, \quad y = t + 2, \quad -2 \leq t \leq 2$$

EXAMPLE 3: Sketch the curve defined by the parametric equations given below:

$$x = \sin t, \quad y = \cos t, \quad 0 \leq t \leq 2\pi$$

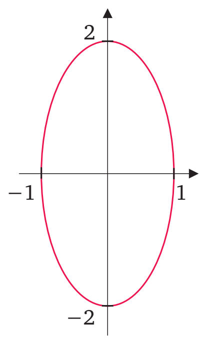
EXAMPLE 4: Match the parametric equations given below with the numbered parametric curves.

a. $x = t, \quad y = t\cos t^2$

b. $x = t, \quad y = \sqrt{1 - t^2}$

c. $x = \sin t, \quad y = 2\cos t$

d. $x = t\sin t, \quad y = t\cos t$



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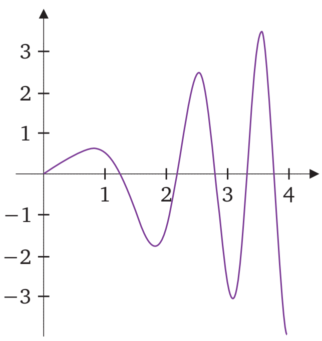

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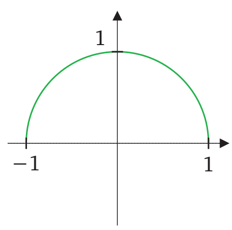

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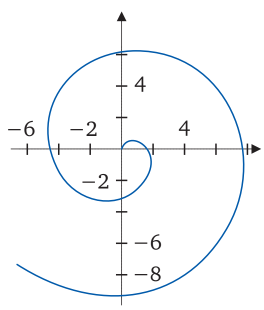


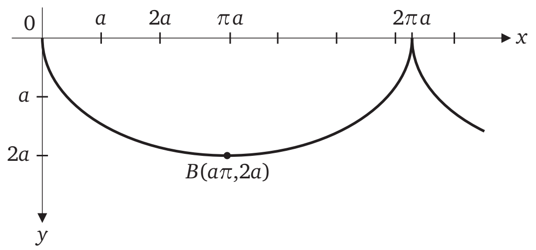
IV

Cycloid

Let us examine the cycloid curve through an example.

EXAMPLE 5: A wheel of radius a rolls without slipping along a horizontal line. Obtain the parametric equations of the path traced by a point P on the wheel that is initially located directly below the center of the wheel. The point P is initially at the origin.

Let us examine the brachistochrone and tautochrone curves.



Calculus with Parametric Curves

Tangent lines and areas

Let us derive the parametric formulas for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

EXAMPLE 6: Find the equation of the tangent line to the ellipse given by the equations $x = 2\cos t$ and $y = 3\sin t$ at the point corresponding to the parameter value $t = \pi/6$.

EXAMPLE 7: For the parametric curve given by $x = t^2 + 1$ and $y = t^3 + t$, find the expression for $\frac{d^2y}{dx^2}$ in terms of t.

EXAMPLE 8: Using an integral, calculate the area of the region enclosed by the curve with parametric equations $x = 2\cos t$ and $y = 2\sin t$ $(0 \leq t \leq \pi/2)$ and the axes.

Length of a parametrically defined curve

Let us examine how to find the length of a parametrically defined curve.

EXAMPLE 9: Calculate the length of the curve $x = e^t \cos t$, $y = e^t \sin t$ on the interval $0 \leq t \leq \pi$.

Surfaces of Revolution

Let us examine how to compute the surface area generated by revolving a parametric curve.

EXAMPLE 10: Calculate the surface area generated by revolving the portion of the curve given by the parametric equations $x = \cos^3 t$ and $y = \sin^3 t$ for $0 \leq t \leq \pi/2$ about the x-axis.

Polar Coordinates

Let us define polar coordinates.

EXAMPLE 11: Find all polar coordinates of the point $P(3, \pi/4)$.

Let us draw the graphs corresponding to the polar equations.

a. $r = 1$ ve $r = -1$

b. $\theta = \pi/4$, $\theta = 5\pi/4$ ve $\theta = -\pi/4$

c. $2 \leq r \leq 4$ ve $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

d. $-2 \leq r \leq 1$ ve $\theta = \frac{\pi}{6}$

e. $\frac{3\pi}{4} \leq \theta \leq \pi$

Let us derive the transformations between polar coordinates and Cartesian coordinates.

EXAMPLE 12: Find the polar equation of the circle given by the equation $(x-2)^2 + y^2 = 4$.

EXAMPLE 13: Convert the polar equations given below into equivalent Cartesian equations and describe their graphs.

a. $r \sin\theta = 5$

b. $r^2 = -6r\sin\theta$

c. $r = \frac{6}{3\cos\theta + 2\sin\theta}$

Graphing Polar Coordinate Equations

Let us examine the symmetry tests in polar coordinates.

I. Symmetry with respect to the x-axis: If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ also lies on the graph.

II. Symmetry with respect to the y-axis: If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ also lies on the graph.

III. Symmetry with respect to the origin: If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ also lies on the graph.

Let us derive the slope of a polar curve given in the form $r = f(\theta)$.

If the curve $r = f(\theta)$ passes through the origin, let us determine its slope at that point.

EXAMPLE 14: Draw the curve given by the equation $r = 1 - \cos\theta$ in the Cartesian xy-plane.

Areas and Lengths in Polar Coordinates

Let us derive the differential area element required for area calculations.

EXAMPLE 15: Find the area of the region in the xy-plane bounded by the cardioid $r = 2(1 + \sin\theta)$.

Let us write a general integral for the area of the region satisfying $0 \le r_1(\theta) \le r \le r_2(\theta)$ and $\alpha \le \theta \le \beta$.

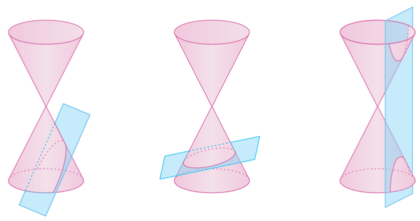
EXAMPLE 16: In the polar coordinate system, consider the region bounded by the cardioid $r_1(\theta) = 2 + 2\cos\theta$ and the circle $r_2(\theta) = 2$. Compute the area of the region that lies outside the circle but inside the cardioid.

Let us examine how to determine the length of a polar curve.

EXAMPLE 17: Calculate the total length of the cardioid given by the equation $r = 1 + \cos\theta$.

Conic Sections

Let us define the parabola, ellipse, and hyperbola geometrically and examine their shapes.



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Parabola

DEFINITION: In a plane, the set of points that are equidistant from a fixed point and a fixed line is called a parabola. This fixed point is called the focus of the parabola, and the fixed line is called the directrix.

EXAMPLE 18: Find the focus and the directrix equation of the parabola given by $x^2 = -12y$.

Ellipse

DEFINITION: An ellipse is the set of points in the plane for which the sum of the distances to two fixed points is constant. These two fixed points are called the foci of the ellipse. The line passing through the foci is called the focal axis, and the midpoint between the foci on this axis is called the center. The points where the focal axis intersects the ellipse are called the vertices.

EXAMPLE 19: For the ellipse given by the equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$, calculate the following quantities:

a. Find the lengths of the semi-major axis (a) and the semi-minor axis (b).

b. Compute the distance from the center to a focus (c).

c. Write the coordinates of the foci of the ellipse.

d. Write the coordinates of the vertices of the ellipse.

e. State the center of the ellipse.

Hyperbola

DEFINITION: A hyperbola is the set of points in the plane for which the absolute difference of the distances to two fixed points (the foci) is constant. The line passing through the foci is called the focal axis. The midpoint between the foci is the center of the hyperbola. The points where the focal axis intersects the hyperbola are called the vertices.

EXAMPLE 20: Find the following characteristic properties of the hyperbola given by $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

a. Center

b. Vertices

c. Foci

d. Asymptotes

Let us discuss what happens if x and y are interchanged in the equations we wrote for the parabola, ellipse, and hyperbola.

Conics in Polar Coordinates

Eccentricity

In conic sections, the eccentricity is denoted by e.

For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a > b$

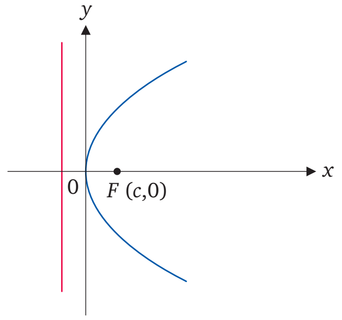
$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a},$$

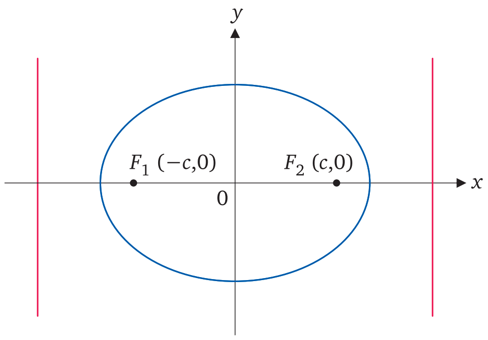
for the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

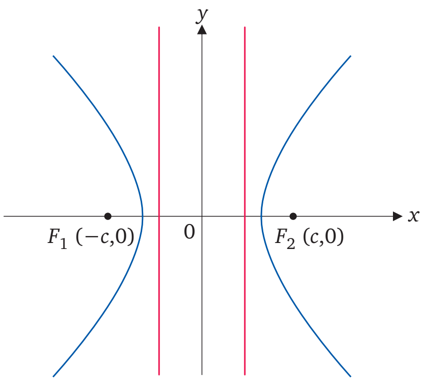
$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a},$$

and for the parabola $e = 1$.

Let us examine the value of eccentricity in detail using the following figures.







EXAMPLE 21: Find the Cartesian equation of the hyperbola whose center is at the origin, one focus is at $(5, 0)$, and the corresponding directrix is $x = \frac{9}{5}$.

Polar equations

Let us see how to derive the polar equation of a parabola, ellipse, or hyperbola.

EXAMPLE 22: Find the standard equation of the hyperbola whose eccentricity is $e = 2$ and whose directrix is the line $x = 3$. The focus of the hyperbola is at the origin.

EXAMPLE 23: Consider the curve whose equation is $r = \frac{12}{3-6\cos\theta}$.

a. Find the eccentricity (e) of this conic section and determine its type.
b. Find the equation of the directrix.
c. Determine the vertices of the curve on the x-axis.

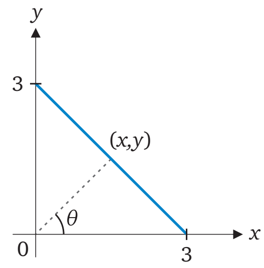
Let us derive the polar equation of an ellipse with eccentricity e and semi-major axis a.

Let us derive the equation of a line in polar coordinates.

Let us derive the equation of a circle in polar coordinates.

Exercises and Problems

1. Parametrize the line segment joining the points $(0,3)$ and $(3,0)$ using the angle θ from the origin as the parameter, as shown in the figure.



2. For the circle $(x+3)^2+(y-1)^2=4$, find a parametrization that starts at the point $(-1,1)$, moves clockwise around the circle for one full revolution, and uses the central angle θ as the parameter.

3. Draw the curve given by the equation $r^2=\sin\theta$ in the Cartesian xy-plane.

4. Compute the area of the region of the circle given by the polar equation $r=4\cos\theta$ for $\pi/6\leq\theta\leq\pi/3$.

5. Show that the equation $x^2 - 4y^2 - 4x + 8y - 4 = 0$ represents a hyperbola. Find the center, the foci, and the asymptotes of this hyperbola.

6. For a conic section with eccentricity $e = \frac{3}{5}$ whose one focus is at the origin of the polar coordinate plane, the corresponding directrix is given by $r\cos\theta = 10$. Find a polar equation for this conic section.

Answers

Example problems

1	a. $x = t, \quad y = t^2, \quad t \in \mathbb{R}$ b. $x = \sqrt{t}, \quad y = t, \quad x \geq 0$ (Right branch)
2	See the video.
3	See the video.
4	a. II b. III c. I d. IV
5	$x = a(\theta - \sin\theta) \qquad y = a(1 - \cos\theta)$
6	$y = -\frac{3\sqrt{3}}{2}x + 6$
7	$\frac{d^2y}{dx^2} = \frac{3t^2-1}{4t^3}$
8	$A = \pi$
9	$L = \sqrt{2}(e^{\pi} - 1)$
10	$S = \frac{6\pi}{5}$
11	$(3, \frac{\pi}{4} + 2n\pi)$ ve $(-3, \frac{5\pi}{4} + 2n\pi)$, $n \in \mathbb{Z}$
12	$r = 4\cos\theta$
13	a. $y = 5$, line b. $x^2 + (y+3)^2 = 9$, circle c. $3x + 2y = 6$, line
14	See the video.
15	$A = 6\pi$
16	$A = 8 + \pi$
17	$L = 8$
18	Focus: $(0, -3)$ Directrix: $y = 3$
19	a. $a = 5$, $b = 3$ b. $c = 4$ c. (4,0) and (−4,0) d. (5,0) and (−5,0) e. (0,0)
20	a. $M(0,0)$ b. $V_1(-3,0)$, $V_2(3,0)$ c. $F_1(-5,0)$, $F_2(5,0)$ d. $y = \pm\frac{4}{3}x$
21	$\frac{x^2}{9} - \frac{y^2}{16} = 1$
22	$r = \frac{6}{1+2\cos\theta}$

23	a. $e=2$, Hyperbola b. $x=-2$ c. $(-4,0)$ and $(-\frac{4}{3},0)$

Exercises and problems

1	$x=\frac{3\cos\theta}{\cos\theta+\sin\theta}$, $y=\frac{3\sin\theta}{\cos\theta+\sin\theta}$, $0\le\theta\le\frac{\pi}{2}$
2	$x=-3+2\cos\theta$, $y=1-2\sin\theta$, $0\le\theta\le 2\pi$
3	See the video.
4	$A=\frac{2\pi}{3}$
5	Center: $(2,1)$ Foci: $(2\pm\sqrt{5},1)$ Asymptotes: $y-1=\pm\frac{1}{2}(x-2)$
6	$r=\frac{30}{5+3\cos\theta}$

CHAPTER 10

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VECTORS & THE GEOMETRY OF SPACE

Motivation

Let us evaluate our motivation for studying this topic.

Three-Dimensional Coordinate Systems

- Let us define the three-dimensional coordinate system to specify a point in space.
- Let us examine the positions of the x, y, and z axes in space.
- Let us see what the right-handed coordinate system represents.
- Let us discuss how the coordinates (x, y, z) of a point are interpreted.
- Let us identify the coordinate planes: the xy-plane, the yz-plane, and the xz-plane.
- Let us evaluate the role of the origin in space.
- Let us see how the coordinate planes divide space.
- Let us define the concept of an octant.
- $x = 1$, $y = 2$, and $z = 4$ planes.

EXAMPLE 1: Determine the geometric regions and objects defined by the following systems of equations and inequalities in the three-dimensional Cartesian coordinate system.

a. $x = 5,\ y = -2$

b. $x \geq 0,\ z = 0$

c. $0 \leq z \leq 3$

d. $x = 0,\ y = 0,\ z \geq 0$

EXAMPLE 2: Describe the geometric object formed by the points in three-dimensional space that simultaneously satisfy the equations $x^2 + z^2 = 9$ and $y = -2$, and determine its center and radius.

Let us express the distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.

Let us examine the standard equation of a sphere with center (x_0, y_0, z_0) and radius a.

EXAMPLE 3: Find the center and radius of the sphere given by the equation $x^2+y^2+z^2-4x+6y-3=0$.

EXAMPLE 4: Describe the geometric regions in $\mathbb{R}^3$ defined by the following equations and inequalities.

a. $x^2+y^2+z^2=9, \quad y \geq 0$

b. $1 < x^2+y^2+z^2 < 16$

c. $x^2+y^2+z^2=25, \quad z=3$

Vectors

Let us discuss why we need the concept of vectors.

DEFINITION: For the vector represented by the directed line segment $\overline{AB}$, the initial point is A and the terminal point is B; its magnitude is denoted by $|\overline{AB}|$. Two vectors are considered equal if they have the same magnitude and the same direction.

DEFINITION: If $\mathbf{v}$ is a two-dimensional vector in the plane equal to a vector whose initial point is the origin and terminal point is (v_1, v_2), then the component form of $\mathbf{v}$ is as follows:

$$\mathbf{v} = \langle v_1, v_2 \rangle$$

If $\mathbf{v}$ is a three-dimensional vector equal to a vector whose initial point is the origin and terminal point is (v_1, v_2, v_3), then the component form of $\mathbf{v}$ is as follows:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

Let us write the vector whose initial point is $P(x_1, y_1, z_1)$ and terminal point is $Q(x_2, y_2, z_2)$, and find its magnitude.

EXAMPLE 5: A vector is given with initial point $P(2,-1,4)$ and terminal point $Q(5,3,-8)$. Accordingly;

a. Find the component form of the vector.

b. Calculate the magnitude of the vector.

Let us define vector addition and scalar multiplication for the vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

EXAMPLE 6: The vectors $\mathbf{u} = \langle 2, -4, 3 \rangle$ and $\mathbf{v} = \langle -1, 5, 2 \rangle$ are given. Find the following:

a. $3\mathbf{u} + 2\mathbf{v}$ b. $\left|\frac{1}{3}\mathbf{u}\right|$ c. $|\mathbf{u} - \mathbf{v}|$

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and a, b be scalars; let us examine the following vector operations.

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- $0\mathbf{u} = \mathbf{0}$
- $1\mathbf{u} = \mathbf{u}$
- $a(b\mathbf{u}) = (ab)\mathbf{u}$
- $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit vectors

Let us define the concept of a unit vector and its basic properties. Let us show the standard unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ in the coordinate plane.

Let us write any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ as a linear combination of the standard unit vectors. Let us obtain the unit vector in the same direction as this vector.

EXAMPLE 7: Find the unit vector $\mathbf{u}$ in the direction of the vector directed from the point $P_1(1, 1, 0)$ to the point $P_2(3, 1, 1)$.

Midpoint of a line segment

Let us examine how we determine the midpoint of the line segment joining the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.

Applications

Let us examine an application of vectors through an example.

EXAMPLE 8: At a construction site, a concrete block weighing 150 N is suspended in a vertical plane by means of two steel cables. The system is in static equilibrium.

a. The first cable makes an angle of 45° to the left of the vertical, and the second cable makes an angle of 30° to the right of the vertical. Calculate the magnitude of the tension force in each cable ($|\mathbf{F}_1|$ and $|\mathbf{F}_2|$).

b. What is the ratio of the magnitude of the total upward force applied by the cables to the magnitude of the weight of the block?

Dot Product

Let us discuss how we can obtain the magnitude of the projection of one vector onto another.

THEOREM: The angle θ between the nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by the following formula:

$$\theta = \arccos\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}\right)$$

DEFINITION: The dot product of the vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is the following scalar value:

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

EXAMPLE 9: $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{k}$ are given. Let θ be the angle between the vectors, and perform the following operations:

a. Calculate the dot product $\mathbf{u} \cdot \mathbf{v}$.

b. Find the magnitudes $|\mathbf{u}|$ and $|\mathbf{v}|$.

c. Find the value of $\cos\theta$.

d. Find the angle θ approximately in degrees.

EXAMPLE 10: A triangle ABC is given with vertices $A=(1,2)$, $B=(4,6)$, and $C=(5,-1)$. Find the angle θ at vertex A.

Orthogonal vectors

Let us write the condition for two nonzero vectors to be perpendicular to each other.

Properties of the dot product

Let us list the algebraic properties of the dot product. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be any vectors and c be a scalar, then

- $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$
- $(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v})=c(\mathbf{u}\cdot\mathbf{v})$
- $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$
- $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^2$
- $\mathbf{0}\cdot\mathbf{u}=0$

Vector projection

Let us define the projection of one vector onto another and express it mathematically.

EXAMPLE 11:
For the vector $\mathbf{u}=2\mathbf{i}+4\mathbf{j}-\mathbf{k}$ onto the vector $\mathbf{v}=\mathbf{i}+\mathbf{j}+\mathbf{k}$;

a. Find its scalar component.

b. Find its vector projection.

EXAMPLE 12: Verify that the vector $\mathbf{u}-\text{proj}_{\mathbf{v}}\mathbf{u}$ is perpendicular to the projection vector $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Work

Let us write the work done by a constant force over a given displacement.

Cross Product

Let us discuss how we can obtain a new vector perpendicular to two vectors by multiplying them.

- Let us write the condition for two vectors to be parallel to each other.
- Let us examine how to obtain the area of a parallelogram using the cross product.

Let us examine the properties of the cross product.
If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any three vectors, and a and b are scalars, then the following properties hold:

- $(a\mathbf{u}) \times (b\mathbf{v}) = (ab)(\mathbf{u} \times \mathbf{v})$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
- $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
- $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
- $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
- $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

Let us examine the cross product of the standard unit vectors with each other.

Let us write the determinant method for the cross product of any two vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

EXAMPLE 13: $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ are given. Accordingly, calculate the following expressions:

a. Find the cross product $\mathbf{u} \times \mathbf{v}$.

b. Find the cross product $\mathbf{v} \times \mathbf{u}$.

EXAMPLE 14: $P(2,-1,3)$, $Q(1,1,-1)$, and $R(-2,0,1)$ are given.

a. Find a vector perpendicular to the plane containing the points P, Q, and R.

b. Calculate the area of the triangle whose vertices are the points P, Q, and R.

Torque

Let us explain the concept of torque and write an equation for this vector.

Scalar triple product or box product

Let us examine the product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ involving three vectors.

Lines and Planes in Space

Lines and line segments in space

Let us examine how a line is represented in space.

- Let us write the vector equation of a line.
- Let us write the parametric equations of a line.
- Let us write the standard equation of a line.
- Let us discuss how a line segment is represented.
- Let us discuss the condition for two lines to be parallel to each other.

EXAMPLE 15: Find the parametric equations of the line passing through the points $P(2,-4,1)$ and $Q(5,0,-3)$.

Let us examine the vector equation of a line through its interpretation as motion in physics.

Let us examine how to calculate the distance between a line and a point.

EXAMPLE 16: Calculate the distance from the point $P(1,1,5)$ to the line whose parametric equations are $x=1+t$, $y=3-t$, and $z=2t$.

Equation of a plane in space

Let us write the equation of the plane that is perpendicular to the nonzero vector $\mathbf{n}=A\mathbf{i}+B\mathbf{j}+C\mathbf{k}$ and passes through the point $P_0(x_0,y_0,z_0)$.

EXAMPLE 17: $A(2,1,1)$, $B(0,4,1)$, and $C(-2,1,4)$ are given.

a. Find a normal vector $\mathbf{n}$ perpendicular to the plane containing these three points.

b. Write the equation of the plane containing these points.

Let us discuss the condition for two planes to be parallel to each other. Let us explain how to find the line of intersection of two non-parallel planes.

EXAMPLE 18: For the two planes given by the equations $x+2y+3z=4$ and $2x-y-z=3$;

a. Find a vector $\mathbf{v}$ parallel to the line of intersection of these two planes.

b. Write the parametric equations of this line of intersection.

EXAMPLE 19: Find the coordinates of the point where the line given by the parametric equations $x=1+t$, $y=2-t$, $z=3+2t$ intersects the plane $x+y+z=10$.

Let us discuss how to find the distance from a point to a plane.

EXAMPLE 20: Find the distance from the point $P(2,4,3)$ to the plane $2x+y+2z=3$.

Let us examine how to find the angle between two planes through an example.

EXAMPLE 21: Find the angle between the planes $x+y=10$ and $y+z=12$.

Cylinders and Quadric Surfaces

Cylinders

Let us examine how a cylinder is formed.

EXAMPLE 22: Find the equation of the cylinder formed by the lines passing through the parabola $x = z^2$, $y = 0$ and parallel to the y-axis.

Quadric surfaces

Let us examine how quadric surfaces are obtained.

EXAMPLE 23: Let us sketch the quadric surface represented by the following equation:

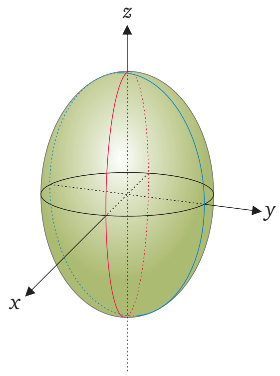
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

EXAMPLE 24: Let us sketch the quadric surface represented by the following equation:

$$4x^2 + 9y^2 + z^2 - 16x + 18y - 11 = 0$$

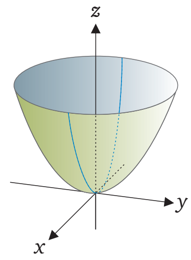
Let us examine some special quadric surfaces.

Elipsoid



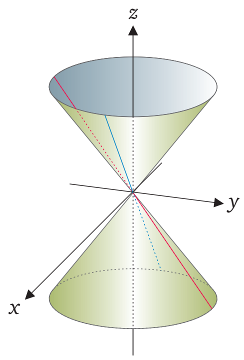
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Eliptik paraboloid



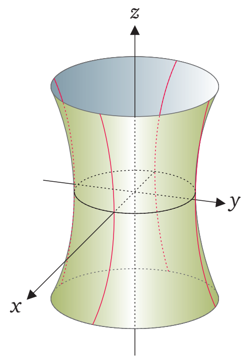
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

Eliptik koni



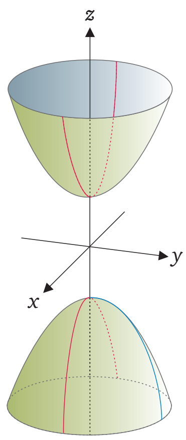
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Tek yapraklı hiperboloid



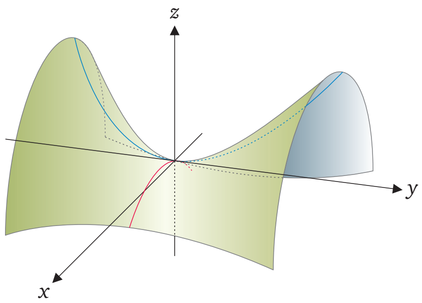
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Çift yapraklı hiperboloid



$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Hiperbolik paraboloid



$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=\frac{z}{c},\quad c>0$$

Exercises and Problems

1. $\mathbf{u}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\mathbf{v}=3\mathbf{i}+4\mathbf{j}$ are given. Accordingly:

a. Calculate the cosine of the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

b. Find the projection vector of $\mathbf{u}$ onto $\mathbf{v}$, namely ($\text{proj}_{\mathbf{v}}\mathbf{u}$).

2. In $\mathbb{R}^3$, the line $\mathscr{L}_1$ is given by parametric equations, while the line $\mathscr{L}_2$ is given by a vector equation:

- $\mathscr{L}_1 : x=1+2t,\quad y=2+t,\quad z=3-t\quad (t\in\mathbb{R})$
- $\mathscr{L}_2 : \mathbf{r}(s)=\langle 2,1,0\rangle+s\langle 1,2,2\rangle\quad (s\in\mathbb{R})$

a. Find the coordinates of the point P where the lines $\mathscr{L}_1$ and $\mathscr{L}_2$ intersect.

b. Calculate the cosine of the acute angle between these two lines.

3. Find the equation of the plane that passes through the point $P(0,1,0)$ and intersects the line $\mathscr{L} : \frac{x-1}{2} = \frac{y-2}{1} = \frac{z+3}{-2}$ perpendicularly. Then calculate the coordinates of the point Q where this plane intersects the line $\mathscr{L}$.

a. Write the equation of the plane.

b. Find the intersection point Q.

4. Show that the points $A(2,1,1)$, $B(3,4,1)$, $C(1,2,3)$, and $D(2,5,3)$ lie on the same plane.

5. The line $\mathscr{L}_1$ passes through the point $(1,2,3)$ and has direction $\mathbf{v}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The line $\mathscr{L}_2$ is given by the parametric equations $x = 2+2t, y = 1-t, z = 3t$. Calculate the shortest distance between these two lines.

6. For $t \in \mathbb{R}$, find the distance between the line given by the parametric equations $x = 1+t,\ y = 2+4t,\ z = 3+t$ and the plane $2x - y + 2z = 10$.

Answers

Your turn questions

1	Go to the video.
2	Go to the video.
3	Center: $(2,-3,0)$, Radius: $r=4$
4	Go to the video.
5	a. $\mathbf{v}=\langle 3,4,-12\rangle$ b. $\lvert\mathbf{v}\rvert=13$
6	a. $\langle 4,-2,13\rangle$ b. $\frac{\sqrt{29}}{3}$ c. $\sqrt{91}$
7	$\mathbf{u}=\frac{2}{\sqrt{5}}\mathbf{i}+\frac{1}{\sqrt{5}}\mathbf{k}$
8	a. $\lvert\mathbf{F}_1\rvert\approx 77.65$ N, $\lvert\mathbf{F}_2\rvert\approx 109.81$ N b. 1
9	a. $\mathbf{u}\cdot\mathbf{v}=14$ b. $\lvert\mathbf{u}\rvert=3$, $\lvert\mathbf{v}\rvert=5$ c. $\cos\theta=\frac{14}{15}$ d. $\theta\approx 21.04^\circ$
10	$\theta=90^\circ$
11	a. $\frac{5\sqrt{3}}{3}$ b. $\frac{5}{3}\mathbf{i}+\frac{5}{3}\mathbf{j}+\frac{5}{3}\mathbf{k}$
12	Go to the video.
13	a. $\mathbf{u}\times\mathbf{v}=3\mathbf{i}+7\mathbf{j}+5\mathbf{k}$ b. $\mathbf{v}\times\mathbf{u}=-3\mathbf{i}-7\mathbf{j}-5\mathbf{k}$
14	a. $\mathbf{n}=\langle 0,2,1\rangle$ b. $A=\frac{7\sqrt{5}}{2}$
15	$x=2+3t$, $y=-4+4t$, $z=1-4t$
16	$d=\sqrt{5}$
17	a. $\mathbf{n}=3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ b. $3x+2y+4z=12$
18	a. $\mathbf{v}=\mathbf{i}+7\mathbf{j}-5\mathbf{k}$ b. $x=2+t$, $y=1+7t$, $z=-5t$
19	$(3,0,7)$
20	$d=\frac{11}{3}$
21	$\theta=60^\circ$ or $\theta=\frac{\pi}{3}$
22	$x=z^2$
23	Go to the video.
24	Go to the video.

Exercises and problems

1	a. $\cos\theta=\frac{2}{15}$ b. $\text{proj}_{\mathbf{v}}\mathbf{u}=\frac{6}{25}\mathbf{i}+\frac{8}{25}\mathbf{j}$
2	a. $P(3,3,2)$ b. $\cos\theta=\dfrac{\sqrt{6}}{9}$
3	a. $2x+y-2z=1$ b. $Q(-1,1,-1)$
4	Go to the video.
5	$d=\frac{5\sqrt{3}}{3}$
6	$d=\frac{4}{3}$

CHAPTER 12

PARTIAL DERIVATIVES - 1

watch video

Motivation

Let us establish our motivation for learning this topic.

Multivariable Functions

- Let us introduce the idea of a function depending on more than one variable.
- Let us think about examples of multivariable functions from daily life.
- Let us discuss what the notation $w = f(x_1, x_2, \ldots, x_n)$ means.
- Let us clarify the distinction between independent and dependent variables.
- Let us examine the concepts of domain and range in this context.

Let us examine the domains and ranges of the functions given below.

- $f(x, y) = \sqrt{16 - x^2 - y^2}$
- $f(x, y) = \ln(x + y)$
- $f(x, y) = \frac{x+y}{x-y}$
- $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$
- $f(x, y, z) = e^{xyz}$
- $f(x, y, z) = xy \ln z$

- Let us define the concepts of interior points and boundary points of a region in the plane.
- Let us express whether a region is open or closed in terms of its boundary points.
- Let us discuss why the inequality $x^2 + y^2 < 1$ describes an open set.
- Let us discuss what it means for a region to be bounded or unbounded through examples.

EXAMPLE 1: $f(x, y) = \ln\left(4 - x^2 - 4y^2\right)$ is given.

a. Find the domain (D) of the function.

b. Describe the domain in the xy-plane and state whether the boundary is included in the domain.

Graphs, level curves, and contour lines of functions of two variables

- Let us consider the basic ways of visualizing functions of two variables.
- Let us discuss how to use level curves and the surface $z = f(x,y)$ to visualize the values of $f(x,y)$.
- Let us examine the difference between level curves and contour lines.

EXAMPLE 2: Draw the graph of the function $f(x,y) = 144 - x^2 - y^2$ and, within its domain in the plane, determine and plot the level curves $f(x,y) = 0$, $f(x,y) = 44$, and $f(x,y) = 108$.

Let us make a similar examination for functions of three variables. Let us examine the level surfaces of the function $f(x,y,z) = \sqrt{x^2+y^2+z^2}$.

Limits and Continuity in Higher Dimensions

DEFINITION: Assume that every open disk centered at (x_0, y_0) contains at least one point of the domain of the function f, other than the point (x_0, y_0) itself. As the point (x,y) approaches (x_0,y_0), we say that the function $f(x,y)$ approaches the number L, and we denote this by

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

This means the following: For every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that for every point (x,y) in the domain of f,

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

whenever

$$|f(x,y) - L| < \varepsilon$$

the inequality holds.

L, M, and k be real numbers. If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$, then:

- $\lim_{(x,y)\to(x_0,y_0)}[f(x,y) + g(x,y)] = L + M$
- $\lim_{(x,y)\to(x_0,y_0)}[f(x,y) - g(x,y)] = L - M$
- $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (for any number k)
- $\lim_{(x,y)\to(x_0,y_0)}[f(x,y) \cdot g(x,y)] = L \cdot M$
- $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \quad M \neq 0$
- $\lim_{(x,y)\to(x_0,y_0)}[f(x,y)]^n = L^n \quad (n \in \mathbb{Z}^+)$
- $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$
  (n is a positive integer; if n is even, it is assumed that $L > 0$.)
- $h(z)$ is continuous at the point $z = L$:
  $\lim_{(x,y)\to(x_0,y_0)} h(f(x,y)) = h(L)$

EXAMPLE 3: Compute the following limits.

a. $\lim_{(x,y)\to(2,1)} \frac{x^2-xy+2}{x^2y+3xy-y^2}$

b. $\lim_{(x,y)\to(0,3)} \sqrt{x^2+y^2+16}$

c. $\lim_{(x,y)\to(0,\pi/2)} \left(\frac{\sin x}{x} + \frac{\cos y}{y-\pi/2}\right)$

d. $\lim_{(x,y)\to(4,4)} \frac{x^2-y^2}{\sqrt{x}-\sqrt{y}}$

EXAMPLE 4: Using the $\varepsilon-\delta$ definition, prove that the limit of the function $f(x,y)=\frac{3x^2y}{x^2+y^2}$ at the point $(x,y)\to(0,0)$ is 0.

Continuity

DEFINITION: Assume that every open disk centered at (x_0, y_0) contains at least one point of the domain of the function f, other than the point (x_0, y_0) itself. In this case, the function $f(x,y)$ is continuous at the point (x_0, y_0) if the following conditions are satisfied:

1. f is defined at the point (x_0, y_0).
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

A function is called continuous if it is continuous at every point in its domain.

EXAMPLE 5: $f(x,y)$ is defined as follows:

$$f(x,y)=\begin{cases} \frac{x^2y^2}{x^4+y^4}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

Show that this function is continuous at points other than the origin, and investigate its continuity at the origin.

Let us show through an example, using the two-path test, that a limit does not exist at a certain point.

EXAMPLE 6: Show that the limit of the function $f(x,y)=\frac{x^3y}{x^6+y^2}$ as $(x,y)\to(0,0)$ does not exist.

EXAMPLE 7: Discuss the continuity of the functions given below:

a. $f(x,y)=e^{x^2+y^2}$

b. $g(x,y)=\cos\left(\frac{x}{y^2+1}\right)$

c. $h(x,y)=\ln\left(4+x^2+y^2\right)$

Partial Derivatives

Let us define the partial derivatives of the function $f(x,y)$ with respect to x and y at the point (x_0, y_0). Let us examine the notation we use for partial derivatives.

Let us examine how a partial derivative is computed through an example.

EXAMPLE 8: $f(x,y) = x^3 - 2x^2y + 4y^2 - 5$ is given.

a. Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

b. Calculate the values of these partial derivatives at the point $(2,-1)$.

EXAMPLE 9: Assuming that the equation $x^2e^z + y\sin z = \ln x + y^2$ defines the variable z as a function of the independent variables x and y, find the partial derivative $\frac{\partial z}{\partial y}$.

Let us examine how a partial derivative is computed for functions of more than two variables through an example.

EXAMPLE 10: Let x, y, and z be independent variables, and suppose the function $f(x,y,z) = z^2 \cdot \cos\left(x^3 + 2y\right)$ is given. Accordingly, calculate the partial derivative $\frac{\partial f}{\partial x}$.

EXAMPLE 11: The function $f(x,y)$ is given as follows:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

a. Find the limit as $(x,y) \to (0,0)$ along the line $y = x$.

b. Find the limit as $(x,y) \to (0,0)$ along the line $y = 0$ (the x-axis).

c. Show that the function f is not continuous at the origin $(0,0)$.

d. Using the definition, show that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at the origin and find their values.

Let us examine second- and higher-order partial derivatives through an example.

EXAMPLE 12: $f(x,y) = x^2 \sin y + y^2 e^x$ is given.

a. Find the first-order partial derivatives of the function ($\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$).

b. Calculate all second-order partial derivatives of the function ($\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, and $\frac{\partial^2 f}{\partial x \partial y}$).

c. Show that the mixed derivatives ($\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$) are equal to each other.

d. Calculate the third-order partial derivatives $\frac{\partial^3 f}{\partial x^2 \partial y}$ and $\frac{\partial^3 f}{\partial y^3}$.

Differentiability

Let us show what the concept of differentiability means for a function. Let us discuss the relationship between the continuity of the function and this concept.

Let us discuss the differentiability of the functions by examining the graphs of two different functions.

- $f(x,y) = x^2 + y^2$
- $g(x,y) = \sqrt{x^2 + y^2}$

Chain Rule

- When both functions $x = x(t)$ and $y = y(t)$ are differentiable with respect to the variable t, let us write the chain rule for a differentiable function $w = f(x,y)$.
- Let us discuss what happens in the case of more than two variables.

Assume that $w = f(x,y,z)$, $x = g(r,s)$, $y = h(r,s)$, and $z = k(r,s)$. If all four of these functions are differentiable, then the function w has partial derivatives with respect to the variables r and s, and they are given by the following formulas:

EXAMPLE 13: $w = x^2y + z^3$ is given. Here, $x = r\cos\theta$, $y = r\sin\theta$, and $z = r^2$. Accordingly, express the partial derivatives $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ in terms of r and θ.

a. $\frac{\partial w}{\partial r}$

b. $\frac{\partial w}{\partial \theta}$

If $w = f(x)$ and $x = g(r,s)$, then the partial derivatives are written as follows:

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}.$$

Let us examine implicit differentiation through an example.

EXAMPLE 14: Calculate the derivative $\frac{dy}{dx}$ for the implicit function given by the equation $e^{xy} + x^2 - y^3 = 5$.

Directional Derivatives and Gradient Vectors

Let us define the gradient vector of the function $f(x,y)$.

Assume that the function $f(x,y)$ is differentiable in an open region containing the point $P_0(x_0, y_0)$, and write the directional derivative of the function at this point in the direction of the unit vector $\mathbf{u}$.

EXAMPLE 15: Find the directional derivative of the function $f(x,y) = y^2\sin(xy) + 3x$ at the point $P(1,0)$ in the direction of the vector $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$.

Let us examine the properties of directional derivatives through an example.

EXAMPLE 16: The function $f(x, y) = \frac{x^3}{3} + \frac{y^2}{2}$ is given. Accordingly;

a. Find the direction in which the function increases most rapidly at the point $(2, 1)$.

b. Find the direction in which the function decreases most rapidly at the point $(2, 1)$.

c. Find the directions in which the change of the function is zero at the point $(2, 1)$.

- Let us examine the relationship between the gradient of a function and its level curves.
- Let us obtain the equation of the line tangent to a level curve of a function.

EXAMPLE 17: Find the equation of the tangent line passing through the point $(-3, 2)$ on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 2$.

Let us examine the algebraic rules for the gradient.

- $\nabla(f + g) = \nabla f + \nabla g$
- $\nabla(f - g) = \nabla f - \nabla g$
- $\nabla(kf) = k\nabla f$
- $\nabla(fg) = f\nabla g + g\nabla f$
- $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Let us also examine directional derivatives in three dimensions through an example.

EXAMPLE 18: For the function $f(x, y, z) = x^2y + \cos(yz) + 2z$ at the point $P_0(2, 0, 1)$;

a. Find the derivative in the direction of the vector $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$.

b. Find the directions in which the function f increases most rapidly and decreases most rapidly at the point P_0, and the rates of change in those directions.

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a smooth curve C, and let $w = f(\mathbf{r}(t))$ be a scalar function evaluated along this curve C. Let us write $\frac{dw}{dt}$ in terms of a gradient and a directional derivative.

Exercises and Problems

1. Let

$$f(x,y) = \begin{cases} \frac{x^2 y \ln(10-x)}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

Find the domain of the function $f(x,y)$ and the set of all points where $f(x,y)$ is continuous.

2. The function

$$f(x,y) = \begin{cases} \frac{2x^2 y + y^3}{x^4+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is given.

a. Show that the limit of $f(x,y)$ does not exist as $(x,y) \to (0,0)$.

b. If they exist, compute $f_x(0,0)$ and $f_y(0,0)$.

c. Write the derivative $f_y(x,y)$ as a piecewise-defined function like $f(x,y)$.

3. A function defined by

$$w = f(u, v) = f(x^2 - y^2, 2xy)$$

is given, where $u = x^2 - y^2$ and $v = 2xy$. It is known that the function f has continuous second-order partial derivatives.

a. Find the derivatives w_x and w_y in terms of f_u and f_v.

b. Write the second-order partial derivative w_{xx} in terms of f_{uu}, f_{uv}, and f_{vv}.

4. The differentiable function $z = z(x, y)$ is defined implicitly by the following equation:

$$z^3 + xy + \ln(z + 1) + y^2 = 6.$$

It is known that $z = 0$ at the point $(x, y) = (1, 2)$.

a. Find the gradient vector of the function $z(x, y)$ at the point $(1, 2)$, that is, $\nabla z(1, 2)$.

b. Find the direction in which the function $z(x, y)$ increases most rapidly at the point $(1, 2)$ and the maximum rate of increase at this point.

Answers

Your turn questions

1	a. $D = \{(x,y) \mid \frac{x^2}{4} + y^2 < 1\}$ b. The interior region of the ellipse $\frac{x^2}{4} + y^2 = 1$, the boundary is not included.
2	Level curves: the circles $x^2 + y^2 = 144$ (radius 12), $x^2 + y^2 = 100$ (radius 10), and $x^2 + y^2 = 36$ (radius 6).
3	a. $\frac{4}{9}$ b. 5 c. 0 d. 32
4	$\delta = \frac{\varepsilon}{3}$
5	See the video.
6	See the video.
7	See the video.
8	a. $\frac{\partial f}{\partial x} = 3x^2 - 4xy, \quad \frac{\partial f}{\partial y} = -2x^2 + 8y$ b. $\left.\frac{\partial f}{\partial x}\right\vert_{(2,-1)} = 20, \quad \left.\frac{\partial f}{\partial y}\right\vert_{(2,-1)} = -16$
9	$\frac{\partial z}{\partial y} = \frac{2y - \sin z}{x^2 e^z + y\cos z}$
10	$\frac{\partial f}{\partial x} = -3x^2 z^2 \sin\left(x^3 + 2y\right)$
11	a. 1/2 b. 0 c. See the video. d. $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$
12	a. $\frac{\partial f}{\partial x} = 2x\sin y + y^2 e^x, \quad \frac{\partial f}{\partial y} = x^2\cos y + 2ye^x$ b. $\frac{\partial^2 f}{\partial x^2} = 2\sin y + y^2 e^x, \quad \frac{\partial^2 f}{\partial y^2} = -x^2 \sin y + 2e^x$ c. $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 2x\cos y + 2ye^x$ d. $\frac{\partial^3 f}{\partial x^2 \partial y} = 2\cos y + 2ye^x, \quad \frac{\partial^3 f}{\partial y^3} = -x^2 \cos y$
13	a. $\frac{\partial w}{\partial r} = 3r^2\cos^2\theta\sin\theta + 6r^5$ b. $\frac{\partial w}{\partial \theta} = r^3\cos^3\theta - 2r^3\cos\theta\sin^2\theta$
14	$\frac{dy}{dx} = \frac{ye^{xy} + 2x}{3y^2 - xe^{xy}}$
15	$\frac{12}{5}$
16	a. $\mathbf{u} = \frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j}$ b. $\mathbf{u} = -\frac{4}{\sqrt{17}}\mathbf{i} - \frac{1}{\sqrt{17}}\mathbf{j}$ c. $\mathbf{n} = \pm\left(-\frac{1}{\sqrt{17}}\mathbf{i} + \frac{4}{\sqrt{17}}\mathbf{j}\right)$
17	$y = \frac{2}{3}x + 4$
18	a. $D_{\mathbf{u}} f(P_0) = \frac{4}{7}$ b. Maximum incr.: $\frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$ in the dir. of $2\sqrt{5}$; maximum decr.: $-\frac{2}{\sqrt{5}}\mathbf{j} - \frac{1}{\sqrt{5}}\mathbf{k}$ in the dir. of $-2\sqrt{5}$

Exercises and problems

1	Domain: $\{(x,y) \in \mathbb{R}^2 \mid x < 10\}$ Set of Continuity: $\{(x,y) \in \mathbb{R}^2 \mid x < 10\} \setminus \{(0,0)\}$
2	a. For the path $y = mx^2$, the limit is $\frac{2m}{1+2m^2}$, depends on m, the limit DNE. b. $f_x(0,0) = 0 \quad f_y(0,0) = \frac{1}{2}$ c. $f_y(x,y) = \begin{cases} \frac{2x^6 + 3x^4y^2 - 4x^2y^2 + 2y^4}{(x^4+2y^2)^2}, & (x,y) \neq (0,0), \\ \frac{1}{2}, & (x,y) = (0,0). \end{cases}$
3	a. $w_x = 2xf_u + 2yf_v, \quad w_y = -2yf_u + 2xf_v$ b. $w_{xx} = 2f_u + 4x^2 f_{uu} + 8xyf_{uv} + 4y^2 f_{vv}$
4	a. $\nabla z(1,2) = -2\mathbf{i} - 5\mathbf{j}$ b. Direction: $-\frac{2}{\sqrt{29}}\mathbf{i} - \frac{5}{\sqrt{29}}\mathbf{j}$, Rate of incr.: $\sqrt{29}$

CHAPTER 12

PARTIAL DERIVATIVES - 2

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Tangent Planes and Differentials

DEFINITION: $f(x,y,z) = c$ at the point P_0 on the level surface, with $\nabla f(P_0) \neq \mathbf{0}$, the tangent plane is perpendicular to the gradient vector, while the normal line is parallel to the gradient vector. In other words, the gradient vector is a normal vector to the surface at that point.

Let us examine how to find the normal vector of a plane defined by $ax + by + cz + d = 0$ using the gradient vector.

EXAMPLE 1: Find the equations of the tangent plane and the normal line to the level surface $f(x,y,z) = x^2 + 2y^2 + 3z^2 - 12 = 0$ at the point $P(1,2,1)$.

EXAMPLE 2: Find the equation of the plane tangent to the surface $f(x,y) = e^x \sin y + x^2 y$ at the point $P(0,\pi,0)$.

EXAMPLE 3: The cylinder $x^2 + y^2 = 5$ and the plane $x + y + z = 1$ intersect along an ellipse E. Find the parametric equations of the line tangent to this ellipse at the point $P(1,2,-2)$.

Let us examine how the value of a differentiable function f changes when we move a small distance ds from the point P_0 in the direction of $\mathbf{u}$.

EXAMPLE 4: Approximately how much does the value of the function $f(x,y,z) = x^2 e^{yz} + \ln(x+z)$ change when moving 0.2 units from the point $P_0(1,0,0)$ toward the point $P_1(3,-2,1)$?

DEFINITION: If the function $f(x,y)$ is differentiable at the point (x_0,y_0), then near this point the function f can be approximated by a linear function. This linear function is

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

and is called the linearization of the function f at the point (x_0,y_0). Accordingly, when the point (x,y) is close to (x_0,y_0), we write

$$f(x,y) \approx L(x,y)$$

This expression gives the linear approximation of the function f at the point in question.

EXAMPLE 5: Find the linearization $L(x,y)$ of the function $f(x,y) = 2x^2 + 3xy - y^2 + 5$ at the point $(1,2)$.

Differentials

DEFINITION: (x_0,y_0) noktasından, bu noktaya yakın $(x_0+dx,\ y_0+dy)$ noktasına geçildiğinde, the linear part of the change in the function f is

$$df = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$$

given by this form. This expression is called the total differential of the function f.

EXAMPLE 6: A metal piece in the shape of a right circular cone is designed with base radius $r = 5$ cm and height $h = 10$ cm. However, due to a manufacturing error, the base radius increased by $dr = 0.1$ cm, while the height changed by $dh = -0.2$ cm. Using differentials, calculate the approximate absolute change in the volume of the cylinder.

Extremum Values and Saddle Points

For a function $f(x, y)$, let us discuss the concepts of local minimum and maximum, as well as absolute minimum and maximum. Let us examine saddle points.

DEFINITION: For an interior point (x_0, y_0) of the domain of the function $f(x, y)$, if $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ or if at least one of the derivatives f_x, f_y is not defined at this point, then the point (x_0, y_0) is called a critical point.

Let us discuss the condition required for a critical point to be a saddle point.

Let us examine how to obtain local extremum points.

THEOREM: Assume that the function $f(x, y)$ and its first- and second-order partial derivatives are continuous throughout a disk centered at (a, b). Also, let $f_x(a, b) = 0$ and $f_y(a, b) = 0$. The Hessian matrix of the function f at the point (a, b) is defined as

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

Let us denote the determinant of this matrix by $D = f_{xx}f_{yy} - {f_{xy}}^2$. Accordingly, the type of the critical point at (a, b) is determined as follows:

- If $D > 0$ and $f_{xx} < 0$, then the function f has a local maximum at the point (a, b).
- If $D > 0$ and $f_{xx} > 0$, then the function f has a local minimum at the point (a, b).
- If $D < 0$, then the function f has a saddle point at the point (a, b).
- If $D = 0$, then the test is inconclusive.

EXAMPLE 7: Find all local extremum values and saddle points of the function $f(x, y) = 4xy - x^4 - 2y^2$.

To find the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region:

I. Find the critical points inside the region and evaluate f.

II. Find the candidate points on the boundary and evaluate f.

III. Compare the obtained values; the largest value is the absolute maximum, and the smallest value is the absolute minimum.

EXAMPLE 8: $f(x, y) = x^2 + 2y^2 - 4x - 4y + 5$ function's absolute maximum and absolute minimum values on the closed triangular region bounded by the lines $x = 0$, $y = 0$, and $x + y = 4$.

EXAMPLE 9: An engineering firm is designing a water tank with a rectangular base and a special constraint at one corner to save space. The sum of the tank's length and the perimeter of its base can be at most 360 cm. Under this constraint, find the dimensions that maximize the volume of the tank and this maximum volume.

Lagrange Multipliers

Let us examine the method of Lagrange multipliers through an example.

EXAMPLE 10: Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

EXAMPLE 11: Find the points on the sphere $x^2+y^2+z^2=9$ that are closest to and farthest from the point $(4,2,-4)$.

EXAMPLE 12: An open-top box in the shape of a rectangular prism will be manufactured using a total of $12\ \text{m}^2$ of cardboard. Find the maximum possible volume of this box.

Let us examine the method of Lagrange multipliers through an example with more than one constraint.

EXAMPLE 13: Find the maximum and minimum values of the function $f(x,y,z)=z$ on the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=5$.

Exercises and Problems

1. $z=2x^2+y^2-3y$ at the point $(2,1,6)$ on the surface given by the equation

a. the tangent plane,

b. the normal line

find the equations.

2. Find the absolute maximum and absolute minimum values of the function $f(x, y) = x^2 + xy + y^2 - 2x - 2y + 1$ on the triangular region R with vertices $A(0,0)$, $B(3,0)$, and $C(0,3)$.

a. Determine the critical points of the function.

b. Find the candidates for extrema on the boundaries of the region.

c. Determine the absolute maximum and absolute minimum values of the function on this region.

3. Find the points on the surface $x^2 + 4yz = 8$ that are closest to the origin.

Answers

Your turn questions

1	Tangent plane: $x+4y+3z=12$ Normal line: $x=1+t,\quad y=2+4t,\quad z=1+3t$
2	$y+z=\pi$
3	$x=1+2t,\quad y=2-t,\quad z=-2-t$
4	$\Delta f \approx df = \frac{7}{15}$ units
5	$L(x,y)=10x-y+1$
6	$dV=\frac{5}{3}\pi$ cm^3
7	Local maximum: $(1,1)$ and $(-1,-1)$ Saddle point: $(0,0)$
8	Absolute maximum: 21, absolute minimum: -1
9	$x=120$ cm, $y=60$ cm, $z=60$ cm $V=432\,000$ cm^3
10	Maximum: 5 Minimum: -5
11	Closest: $(2,1,-2)$, Farthest: $(-2,-1,2)$
12	$V=4.0$ m^3
13	Maximum: $5+\sqrt{2}$, Minimum: $5-\sqrt{2}$

Exercises and problems

1	a. $-8x+y+z+9=0$ b. $x=2-8t,\quad y=1+t,\quad z=6+t$
2	a. $\left(\frac{2}{3},\frac{2}{3}\right)$ b. $(1,0),(0,1),\left(\frac{3}{2},\frac{3}{2}\right),(0,0),(3,0),(0,3)$ c. Absolute maximum: 4, Absolute minimum: $-\frac{1}{3}$
3	$(0,\sqrt{2},\sqrt{2})$ and $(0,-\sqrt{2},-\sqrt{2})$

CHAPTER 13

MULTIPLE INTEGRALS

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Double and Iterated Integrals over Rectangular Regions

Let us examine the relationship between the double integral and volume.

THEOREM: If the function $f(x,y)$ is continuous throughout the rectangular region $R: a \le x \le b, c \le y \le d$, then:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

EXAMPLE 1: Find the volume of the solid lying above the region in the xy-plane bounded by the rectangle $R: 0 \le x \le 2$, $0 \le y \le 1$ and below the elliptic paraboloid $z = 4 + x^2 + 2y^2$.

Double Integrals over Nonrectangular Regions

Let us examine how to evaluate double integrals over nonrectangular regions through an example.

EXAMPLE 2: Find the volume of the solid that lies above the triangular region in the xy-plane bounded by the lines $x = 0$, $y = 2$, and $y = x$, and whose upper surface is the plane $z = f(x,y) = 3x + y$.

EXAMPLE 3: Evaluate the following double integral by changing the order of integration:

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$$

EXAMPLE 4: Let D be the region bounded by the curves $y=\sqrt{x}$, $y=0$, and $x+y=2$. The integral $\iint_D f(x,y)\,dA$ will be written with its limits.

a. Write the integral in the order $dx\,dy$.

b. Write the integral in the order $dy\,dx$.

If the functions $f(x,y)$ and $g(x,y)$ are continuous on a bounded region R, then the following properties hold:

- For any number c:

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$$

- Sum and difference rule:

$$\iint_R \big(f(x,y)\pm g(x,y)\big)\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

- If $f(x,y)\ge 0$ on R:

$$\iint_R f(x,y)\,dA \ge 0$$

- If $f(x,y)\ge g(x,y)$ on R:

$$\iint_R f(x,y)\,dA \ge \iint_R g(x,y)\,dA$$

- If the region R is the union of two nonoverlapping regions R_1 and R_2:

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

EXAMPLE 5: Find the volume of the solid that lies in the first octant and is bounded by the surface $z=1-y^2-x$.

Area Calculation Using a Double Integral

Let us examine how to calculate area using a double integral.

EXAMPLE 6: We want to calculate the area of the region R in the first quadrant bounded by the curves $y = \sqrt{x}$ and $y = x^3$ using a double integral. Without evaluating the integrals that give the area of this region, express them in two different forms:

a. with the order of integration $dy\,dx$,

b. with the order of integration $dx\,dy$

Let us express the average value of a function $f(x, y)$ over a region R using a double integral.

Double Integrals in Polar Coordinates

Let us examine how the area of a region R is expressed by a double integral in polar coordinates.

EXAMPLE 7: In the xy-plane, let R denote the region that lies inside the circle $r = 2\cos\theta$ and outside the circle $r = 1$. Calculate the area of the region R.

EXAMPLE 8: Evaluate the integral $\iint_R e^{x^2+y^2}\,dy\,dx$. Here, R is the semicircular region lying between the x-axis and the curve $y=\sqrt{1-x^2}$.

EXAMPLE 9: Find the volume of the part of the solid lying between the surface $z=x^2+y^2+4$ and the xy-plane whose base in the xy-plane is the annular region between the circles $x^2+y^2=1$ and $x^2+y^2=9$.

EXAMPLE 10: Using the polar integration method, find the area of the region R bounded by the circle $x^2+y^2=4$, lying above the line $y=1$ and below the line $y=x$.

Triple Integrals in Cartesian Coordinates

Let us examine how volume is calculated using a triple integral.

EXAMPLE 11: Let S be the sphere centered at the origin with radius 4. Let D denote the solid region inside this sphere and above the plane $z=2$. For a function $F(x,y,z)$ defined on the region D,

$$\iiint_D F(x,y,z)\,dV$$

set up the integral in Cartesian coordinates in the order $dz\,dy\,dx$.

EXAMPLE 12: Write the volume of the tetrahedron D with vertices $O(0,0,0)$, $A(2,1,0)$, $B(0,1,0)$, and $C(0,1,2)$ using a triple integral, taking the order of integration as $dz\,dy\,dx$.

EXAMPLE 13: Write the triple integral that gives the volume of the region D bounded below by the paraboloid $z=x^2+2y^2$ and above by the paraboloid $z=10-x^2-y^2$, in the order $dz\,dy\,dx$.

Applications

Let us examine how to calculate the mass of solid objects using multiple integrals. Let us write how to find the center of mass of the object.

EXAMPLE 14: The density of the solid that rises above the disk $x^2 + y^2 \leq 9$ in the xy-plane and is bounded above by the paraboloid $z = 9 - x^2 - y^2$ is given as $\delta(x, y, z) = z$.

a. Write the triple integral that gives the total mass of this solid in Cartesian coordinates.

b. Write the integrals that give the x, y, and z coordinates of the center of mass of the solid in Cartesian coordinates.

Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

- Let us define cylindrical coordinates.
- Let us write the necessary equations for cylindrical coordinates.
- Let us examine the surfaces represented by the equations $r = 2$, $\theta = \frac{\pi}{2}$, and $z = 1$.

Let us show how triple integrals are evaluated in cylindrical coordinates.

EXAMPLE 15: The region D is bounded above by the paraboloid $z = 4 - x^2 - y^2$, below by the plane $z = 0$, and laterally by the cylinder $x^2 + y^2 = 2x$. Using cylindrical coordinates, determine the limits of the triple integral of a general function $f(r, \theta, z)$ defined on this region and write the integral.

EXAMPLE 16: Calculate the volume of the region between the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane using cylindrical coordinates.

Spherical Coordinates

- Let us define spherical coordinates.
- Let us write the necessary equations for spherical coordinates.
- Let us examine the surfaces represented by the equations $\rho = 2$, $\phi = \frac{\pi}{2}$, and $\theta = \frac{\pi}{3}$.

EXAMPLE 17: Express the equation of the sphere $x^2 + y^2 + z^2 = 4z$ in spherical coordinates (ρ, ϕ, θ) and determine the Cartesian coordinates of the center of this surface.

Let us show how triple integrals are evaluated in spherical coordinates.

EXAMPLE 18: Using the spherical coordinate system, calculate the volume of a sphere of radius a with the help of a triple integral.

EXAMPLE 19: Write the integral that gives the volume of the ice-cream-cone-shaped region D bounded above by the sphere $\rho = 2$ and below by the cone $\phi = \pi/6$ using spherical coordinates.

Change of Variables in Multiple Integrals

Let us examine what happens when we transform the region and the variables in double integrals. Let us define the Jacobian determinant.

EXAMPLE 20: Find the Jacobian determinant for the polar coordinate transformation $x = r\cos\theta,\ y = r\sin\theta$, and use it to express the Cartesian integral $\iint_R f(x,y)\,dx\,dy$ as a polar integral.

EXAMPLE 21: In the xy-plane, the region R is bounded by the curves $y = \frac{1}{x}$, $y = \frac{4}{x}$, $y = x^2$, and $y = 4x^2$. Calculate the area of this region.

EXAMPLE 22: If the region R is the trapezoidal region with vertices at $(1,0),(2,0),(0,1)$, and $(0,2)$, use the transformations $u = x - y$ and $v = x + y$ to evaluate the integral

$$\iint_R \exp\left(\frac{x-y}{x+y}\right) dA$$

.

a. Find the absolute value of the determinant of the Jacobian matrix of the transformation, that is, the expression $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$.

b. Determine the new region of integration in the uv-plane and evaluate the integral.

Let us examine the Jacobian matrix for the transformation $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$ in three dimensions:

$$J(\rho,\phi,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2 \sin\phi$$

EXAMPLE 23: Write the following triple integral over an appropriate region in uvw-space using the transformations $u = x + y + z$, $v = x - y$, and $w = x$:

$$\int_0^2 \int_0^x \int_0^{2-x} \left(\frac{x+y+z}{2}\right) dz\, dy\, dx$$

Exercises and Problems

1. Below, the sum of two double integrals defined over two different regions is given:

$$\int_0^1 \int_0^{x^2} f(x,y)\, dy\, dx \; + \int_1^3 \int_0^{\frac{3-x}{2}} f(x,y)\, dy\, dx.$$

Combine the regions defined by these two integrals in the xy-plane. By changing the order of integration over the resulting combined region, write the total integral as a single double integral.

2. Evaluate the following double integral:

$$\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \cos\left(y^2\right) dy\, dx$$

3. Let R be the region between the line $y = x$ and the parabola $y = x^2$. Calculate the average value of the product of the coordinates of the points over the region R.

4. Using the integral given in cylindrical coordinates

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta, \quad r \geq 0$$

perform the following operations:

a. Convert the integral to Cartesian coordinates in the order $dz\, dy\, dx$.

b. Convert the integral to spherical coordinates.

c. Find the result by evaluating one of the integrals.

5. Let R be the closed region in the first quadrant of the xy-plane bounded by the curves $xy=1$, $xy=4$ and the lines $y=x$, $y=4x$. Accordingly, evaluate the following double integral:

$$\iint_R y^2\,dA$$

Answers

Your turn questions

1	$V=12$
2	$V=\frac{20}{3}$
3	$\frac{1}{2}(e-1)$
4	a. $\int_0^1\int_{y^2}^{2-y} f(x,y)\,dx\,dy$ b. $\int_0^1\int_0^{\sqrt{x}} f(x,y)\,dy\,dx+\int_1^2\int_0^{2-x} f(x,y)\,dy\,dx$
5	$V=\frac{4}{15}$
6	a. $A=\int_0^1\int_{x^3}^{\sqrt{x}}dy\,dx$ b. $A=\int_0^1\int_{y^2}^{y^{1/3}}dx\,dy$
7	$A=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$
8	$\frac{\pi(e-1)}{2}$
9	$V=72\pi$
10	$A=\frac{\pi}{6}+\frac{1-\sqrt{3}}{2}$
11	$\int_{-2\sqrt{3}}^{2\sqrt{3}}\int_{-\sqrt{12-x^2}}^{\sqrt{12-x^2}}\int_2^{\sqrt{16-x^2-y^2}} F(x,y,z)\,dz\,dy\,dx$
12	$V=\int_0^2\int_{x/2}^1\int_0^{2y-x}dz\,dy\,dx$
13	$V=\int_{-\sqrt{5}}^{\sqrt{5}}\int_{-\sqrt{(10-2x^2)/3}}^{\sqrt{(10-2x^2)/3}}\int_{x^2+2y^2}^{10-x^2-y^2}dz\,dy\,dx$
14	a. $M=\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} z\,dz\,dy\,dx$ b. $\bar{x}=\frac{1}{M}\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} xz\,dz\,dy\,dx,$ $\bar{y}=\frac{1}{M}\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} yz\,dz\,dy\,dx,$ $\bar{z}=\frac{1}{M}\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} z^2\,dz\,dy\,dx$
15	$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^{4-r^2} f(r,\theta,z)\,r\,dz\,dr\,d\theta$
16	$V=8\pi$
17	$\rho=4\cos\phi$ Center: $(0,0,2)$
18	$V=\frac{4}{3}\pi a^3$
19	$V=\int_0^{2\pi}\int_0^{\pi/6}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$
20	Jacobian determinant: r Integral: $\iint_G f(r\cos\theta,r\sin\theta)r\,dr\,d\theta$
21	$\ln(4)$
22	a. $\left\lvert\frac{\partial(x,y)}{\partial(u,v)}\right\rvert=\frac{1}{2}$ b. $\frac{3}{4}(e-e^{-1})$
23	$\int_0^2\int_0^w\int_{2w-v}^{2+w-v}\frac{u}{2}\,du\,dv\,dw$

Exercises and problems

1	$\int_0^1\int_{\sqrt{y}}^{3-2y} f(x,y)\,dx\,dy$
2	$\frac{1}{2}$
3	$\frac{1}{4}$
4	a. $\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}}dz\,dy\,dx$ b. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ c. $\frac{8\pi}{3}(2-\sqrt{2})$
5	$\frac{45}{4}$

CHAPTER 14

INTEGRALS AND VECTOR FIELDS

watch video

Line Integrals of Scalar Functions

Let us discuss what the integral of a scalar function over a curve means and examine how this integral is evaluated.

EXAMPLE 1: Integrate the function $f(x,y,z) = x + y - 3z^2$ along the line segment C connecting the origin to the point $(1,2,1)$.

Let us demonstrate the additivity of line integrals through an example.

EXAMPLE 2: Evaluate the integral of $f(x,y,z) = x + y + z$ over the path $C = C_1 \cup C_2$, where C_1 is the line segment from $(0,0,0)$ to $(1,0,0)$, and C_2 is the line segment from $(1,0,0)$ to $(1,1,1)$.

Let us see how to compute the mass of a wire-shaped object through an example.

EXAMPLE 3: A thin metal wire lies along the upper half of the circle $x^2 + y^2 = 4$ where $y \geq 0$, placed in the xy-plane. The linear density at a point (x,y) on the wire is given by the function $\delta(x,y) = 3 - y$.

a. Find the total mass of the wire.

b. Write the integral expressions that give the coordinates of the center of mass of the wire.

Vector Fields and Line Integrals

- Let us examine the concept of a vector field.
- Let us examine the vectors $\mathbf{T}$ and $\mathbf{N}$ that can be defined on a curve $\mathbf{r}(t)$.
- Let us examine the gradient field, which describes the rate of change of a scalar function $f(x,y,z)$.

DEFINITION: Let C be a smooth curve parametrized by the position vector $\mathbf{r}(t)$, $a \leq t \leq b$. Let $\mathbf{F}$ be a vector field defined along C with continuous components. Then the line integral of the vector field $\mathbf{F}$ along the curve C

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\right)ds = \int_C \mathbf{F}\cdot d\mathbf{r}$$

is defined as above, where $\mathbf{T}$ denotes the unit tangent vector to the curve C and s denotes the arc length parameter.

After solving the following example, let us discuss the work done by a force field and the concepts of flux integral and circulation for a velocity field.

EXAMPLE 4: Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ of the vector field $\mathbf{F}(x,y,z) = y\mathbf{i}+z\mathbf{j}+x^2\mathbf{k}$ along the curve C given by $\mathbf{r}(t) = t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$ for $0 \leq t \leq 1$.

Let us show that for the vector field $\mathbf{F}(x,y,z) = M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$, the equality

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy + P\,dz$$

holds.

DEFINITION: Let C be a smooth, simple closed curve lying in the domain of a continuous vector field $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ defined in the plane. If $\mathbf{n}$ is the outward-pointing unit normal vector on C, then the outward flux of $\mathbf{F}$ along C is defined as follows:

$$\text{Flux} = \int_C \mathbf{F}\cdot\mathbf{n}\,ds$$

Let us rewrite the same flux by expressing the vector $\mathbf{n}$ in a different form and obtain the following result:

$$\text{Flux} = \oint_C M\,dy - N\,dx$$

EXAMPLE 5: Given the vector field $\mathbf{F} = (2x + y)\mathbf{i} + (y - x)\mathbf{j}$, find the outward flux across the circle $x^2 + y^2 = 4$ traversed once in the positive direction in the xy-plane.

Path Independence, Conservative Fields, and Potential Functions

- Let us define the concept of path independence.
- Let us define the concept of a conservative field.
- Let us define the concept of a potential function.

Let us write the vector calculus counterpart of the fundamental theorem of calculus. Let us state the fundamental theorem for line integrals.

EXAMPLE 6: The force field $\mathbf{F} = \nabla f$ is given as the gradient of the potential function

$$f(x, y, z) = -\frac{1}{x^2 + y^2 + z^2}.$$

An object moves along a smooth curve C that starts at $(1, 0, 0)$, passes through $(1, 1, 1)$, ends at $(2, 2, 1)$, and does not pass through the origin. Find the work done by the force field $\mathbf{F}$ during this motion.

THEOREM: Let the components of the vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be continuous on an open, connected region D in space. The field $\mathbf{F}$ is conservative if and only if there exists a differentiable function f defined on D such that

$$\mathbf{F} = \nabla f.$$

In this case, the function f is called the potential function of the field $\mathbf{F}$.

Let us examine the closed-path property of conservative fields.

Let the vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be defined on an open, simply connected region. Assume further that the components M, N, and P have continuous first-order partial derivatives.

Under these conditions, let us write the condition required for the field $\mathbf{F}$ to be conservative.

EXAMPLE 7: Show that the vector field $\mathbf{F} = (e^x \sin y + yz)\mathbf{i} + (e^x \cos y + xz + z)\mathbf{j} + (xy + y)\mathbf{k}$ is conservative on its natural domain and find a potential function for this field.

EXAMPLE 8: Consider the following vector field:

$$\mathbf{F} = \left(\frac{-y}{x^2+y^2}\right)\mathbf{i} + \left(\frac{x}{x^2+y^2}\right)\mathbf{j} + (z+1)\mathbf{k}$$

Answer the following questions about this vector field:

a. Show that the vector field satisfies the component test for conservative fields.

b. Determine the natural domain of the vector field $\mathbf{F}$ and explain why this domain is not simply connected.

c. Evaluate the integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ along the unit circle in the xy-plane. Interpret whether the vector field is conservative in light of this result.

Let us examine the concept of *exact differential form.*

EXAMPLE 9: Show that the expression $z\,dx + 2y\,dy + x\,dz$ is an exact differential form and evaluate the following integral along any curve from $(1,1,1)$ to $(3,2,2)$:

$$\int_{(1,1,1)}^{(3,2,2)} z\,dx + 2y\,dy + x\,dz$$

Green's Theorem in the Plane

DEFINITION: The circulation density of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the following scalar expression:

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Let us examine the curl or tangential form of Green's theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

EXAMPLE 10: Verify the curl form of Green's theorem for the vector field $\mathbf{F}(x, y) = (x^2 - y)\mathbf{i} + (x + y^2)\mathbf{j}$ and the unit disk region R with boundary
$C : \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi$, by evaluating the following integrals:

a. Evaluate the circulation integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

b. Evaluate the double integral $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$.

DEFINITION: The divergence of the vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is given by:

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Let us examine the divergence or normal form of Green's theorem:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dx\,dy$$

EXAMPLE 11: Verify the divergence form of Green's theorem for the vector field $\mathbf{F}(x,y)=(x^2-y)\mathbf{i}+(x+y^2)\mathbf{j}$ and the unit disk region R with boundary $C:\mathbf{r}(t)=(\cos t)\mathbf{i}+(\sin t)\mathbf{j},\ 0\le t\le 2\pi$, by evaluating the following integrals:

a. Evaluate the flux integral $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$.

b. Evaluate the double integral $\iint_R\left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}\right)dA$.

EXAMPLE 12: Let C be the counterclockwise boundary of the unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. Evaluate the following line integral:

$$\oint_C (e^x+3y^2)\,dx+(4x+\sin y)\,dy$$

Surfaces and Area

Let us understand the parametrization of surfaces through an example.

EXAMPLE 13: Find appropriate parametrizations and parameter ranges for the surfaces defined below:

a. The portion of the cone $z=\sqrt{x^2+y^2}$ lying in the interval $0\le z\le 1$.

b. The entire sphere $x^2+y^2+z^2=a^2$.

c. The portion of the cylinder $x^2+(y-2)^2=4$ lying in the interval $0\le z\le 5$.

Let us examine how to derive the integral that gives the area of a smooth parametric surface defined as $\mathbf{r}(u,v)=f(u,v)\mathbf{i}+g(u,v)\mathbf{j}+h(u,v)\mathbf{k}$ for $a\le u\le b$ and $c\le v\le d$.

EXAMPLE 14: Find the surface area of the cone given by the parametrization $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$ over the intervals $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

Let us derive the surface area differential $d\sigma$ for the surface $z = f(x,y)$ defined over a region R in the xy-plane. Let us also derive the same for the surface $F(x,y,z) = c$.

EXAMPLE 15: Write the integral in Cartesian coordinates that gives the surface area of the portion of the paraboloid $x^2 + y^2 - z = 0$ lying below the plane $z = 1$.

Surface Integrals

Let us write the surface integral of a function $G(x,y,z)$ over the surface S according to the following representations of S:

- Parametrically defined surfaces.
- Surfaces defined explicitly as $z = f(x,y)$.
- Surfaces defined implicitly as $F(x,y,z) = c$.

EXAMPLE 16: Let S be the triangular surface in the first octant lying on the plane $x+y+z=1$. Write the surface integral of $G(x,y,z)=z$ over S using the following three different methods:

a) By parametrizing the surface S.

b) By writing the surface S explicitly as $z=f(x,y)$.

c) By treating the surface S implicitly as $F(x,y,z)=c$.

EXAMPLE 17: Consider the surface S on the cylinder $z=x^2+1$ over the triangular region R in the xy-plane defined by $x\geq 0$, $y\geq 0$, and $x+y\leq 1$. Set up the double integral in the order $dy\,dx$ needed to evaluate the surface integral $\iint_S \sqrt{y(1+4x^2)}\,d\sigma$ over this surface.

Surface Integrals of Vector Fields

Let us examine the concept of an orientable surface.

DEFINITION: Let $\mathbf{F}$ be a vector field defined in space, and let S be a smooth surface with a chosen unit normal vector field $\mathbf{n}$. Then the surface integral of the vector field $\mathbf{F}$ over the surface S is defined as:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$$

This integral is also referred to as the flux of the vector field $\mathbf{F}$ across the surface S.

Let us write the flux of a vector field $\mathbf{F}$ across the surface S according to the following representations of S:

- Parametrically defined surfaces.
- Surfaces defined implicitly as $g(x,y,z)=c$.
- Surfaces defined explicitly as $z=f(x,y)$.

EXAMPLE 18: Let $\mathbf{F} = 2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ and let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ lying above the xy-plane. Write the double integral in terms of x and y, with explicit limits, that gives the upward flux across S.

Stokes' Theorem

Let us define the curl of a vector field.

Let S be a piecewise smooth, oriented surface with boundary curve C. Let us write Stokes' theorem for the counterclockwise circulation of the vector field $\mathbf{F}$ along the boundary C.

Let us rewrite Stokes' theorem for a counterclockwise-oriented curve in the xy-plane.

EXAMPLE 19: Let $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and let S be the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$. The boundary curve of this surface is the circle $C : x^2 + y^2 = 4$, $z = 0$, oriented counterclockwise when viewed from above. Verify Stokes' Theorem by evaluating both sides of the equation separately.

a. Evaluate the circulation integral along the curve C.

b. Evaluate the curl integral $\nabla \times \mathbf{F}$ over the surface.

Let us show how we can isolate the curl component at a point and obtain the circulation density using Stokes' theorem.

EXAMPLE 20: The velocity field of a fluid is given by $\mathbf{F} = -2y\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$, which rotates around the z-axis while also accelerating upward in the z-direction.

a. Find the curl $\nabla \times \mathbf{F}$.

b. Use Stokes' Theorem to compute the circulation of $\mathbf{F}$ along the circle C of radius ρ centered at $(0, 0, 3)$ in the plane $z = 3$, oriented counterclockwise when viewed from above. Relate the result to the circulation density.

Let us discuss Stokes' theorem for surfaces with holes.

Let us show that $\nabla \times \nabla f = \mathbf{0}$. Let us state what this identity means for conservative vector fields.

The Divergence Theorem

Let us define the divergence of a vector field.

Let $\mathbf{F}$ be a vector field with continuous first-order partial derivatives, and let S be a piecewise smooth, oriented closed surface. Let us write the Divergence Theorem for the flux of $\mathbf{F}$ across S in the direction of the outward unit normal field $\mathbf{n}$.

EXAMPLE 21: Let $D : x^2 + y^2 + z^2 \leq 1$ be the unit ball and let $S : x^2 + y^2 + z^2 = 1$ be the outward-oriented sphere bounding D. Verify the Divergence Theorem for the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ by evaluating the following two integrals.

a. Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ giving the outward flux.

b. Evaluate the volume integral $\iiint_D \nabla \cdot \mathbf{F}\, dV$.

Let us show that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. Let us state what this identity means in the context of the Divergence Theorem.

Let us discuss the Divergence Theorem for regions that can be subdivided into parts.

Let us examine the Divergence Theorem for a vector field that points away from the origin and weakens inversely with the square of the distance. We will use a spherical shell region with inner radius b and outer radius a as our reference.

Let us compare the result we obtain here with Gauss's law used in electromagnetic theory.

Let D be a region in space bounded by a closed, oriented surface S. Let $\mathbf{v}(x,y,z)$ be the velocity field of a fluid flowing smoothly through D, let $\delta = \delta(t,x,y,z)$ be the fluid density at the point (x,y,z) at time t, and let $\mathbf{F} = \delta\mathbf{v}$.

Let us derive the continuity equation of hydrodynamics using the Divergence Theorem with the quantities defined above.

Let us regard the two-dimensional vector field $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ as a three-dimensional field whose $\mathbf{k}$-component is zero. Using this,

- let us re-derive the curl or tangential form of Green's theorem.
- let us re-derive the divergence or normal form of Green's theorem.

NOTE: The integral of a differential operator acting on a field over a region equals the sum of the appropriate field components on the boundary of that region.

Exercises and Problems

1. Find the work done by the force field $\mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ along the plane curve $\mathbf{r}(t) = \left(e^{2t}\cos t\right)\mathbf{i} + \left(e^{2t}\sin t\right)\mathbf{j}$ from $(1,0)$ to $(-e^{2\pi},0)$ using two methods:

a. By using the parametrization of the curve to evaluate the work integral.

b. By finding a potential function for $\mathbf{F}$ and using it.

2. Let $\mathbf{F} = (x^2+z)\mathbf{i} + (y^2+2x)\mathbf{j} + (z^2-y)\mathbf{k}$ be a vector field. Let C be the closed curve formed by the intersection of the sphere $x^2+y^2+z^2=1$ and the cone $z=\sqrt{x^2+y^2}$.

The curve C is oriented counterclockwise around the z-axis when viewed from above. Find the circulation of $\mathbf{F}$ along C.

3. Using Green's Theorem, show that the area of a region R enclosed by a simple closed curve C oriented counterclockwise in the plane can be computed by the following line integral:

$$\text{Area}(R) = \frac{1}{2}\oint_C x\,dy - y\,dx$$

4. Let $\mathbf{F} = y^2\mathbf{i} - yz\mathbf{j}$ and let S be the surface bounded by the parabolic cylinder $x = y^2$ for $-1 \le y \le 1$ and the planes $z = 0$ and $z = 3$. Given that the orientation of S points away from the xz-plane, compute the flux of $\mathbf{F}$ across S ($\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$).

Answers

Your turn questions

1	$\frac{\sqrt{6}}{2}$
2	$\frac{1}{2} + 2\sqrt{2}$
3	a. $M = 6\pi - 8$ b. $\bar{x} = \frac{4}{M}\int_0^{\pi} \cos\theta(3 - 2\sin\theta)d\theta$ $\bar{y} = \frac{4}{M}\int_0^{\pi} \sin\theta(3 - 2\sin\theta)d\theta$
4	$\frac{4}{3}$
5	$\text{Flux} = 12\pi$
6	$\frac{8}{9}$
7	$f(x,y,z) = e^x \sin y + xyz + yz + C$
8	a. See the video. b. $D = \mathbb{R}^3 - \{z\text{-axis}\}$ c. 2π, not conservative.
9	8
10	a. 2π b. 2π
11	a. 0 b. 0
12	1
13	a. $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1, 0 \le \theta \le 2\pi$ b. $\mathbf{r}(\phi,\theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j}$ $+(a\cos\phi)\mathbf{k}, \quad 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$ c. $\mathbf{r}(\theta,z) = (2\sin 2\theta)\mathbf{i} + (4\sin^2\theta)\mathbf{j} + z\mathbf{k}$, $0 \le \theta \le \pi, 0 \le z \le 5$
14	$\pi\sqrt{2}$
15	$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$
16	$\int_0^1\int_0^{1-x} (1 - x - y)\sqrt{3}\, dy\, dx$
17	$\int_0^1\int_0^{1-x} \sqrt{y}(1 + 4x^2)\, dy\, dx$
18	$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4x + 2xy^2 + 4 - x^2 - y^2\right) dy\, dx$
19	a. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi$ b. $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 4\pi$
20	a. $\nabla \times \mathbf{F} = 4\mathbf{k}$ b. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 4\pi\rho^2$ and $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \frac{1}{\pi\rho^2}\oint_C \mathbf{F} \cdot d\mathbf{r} = 4$
21	a. 4π b. 4π

Exercises and problems

1	a. $e^{2\pi} - 1$ b. $f(x,y) = \sqrt{x^2 + y^2}$
2	π
3	See the video.
4	8