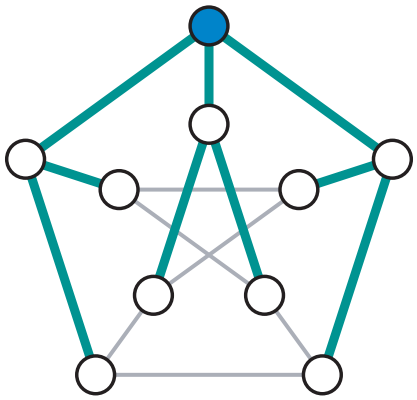
DISCRETE MATHEMATICS

PROBLEM BOOK

SAMPLE PAGES

13 chapters · 180 problems · step-by-step solutions



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How to use this book

- Each chapter opens with a reminder box summarizing the definitions and formulas the problems rely on.
- Difficulty: ★☆☆ basic ★★☆ intermediate ★★★ advanced. Each chapter has a few multiple-choice concept questions; each chapter ends with exam-style mixed problems.
- The dotted area below each problem is for your own work. Solve first; if stuck, go back to the reminder box and the lecture notes, and look at the solution last.
- All solutions are at the end of the book. The Solution → link next to a problem takes you to its solution, and ← Problem takes you back. Short answers are in the **ANSWER** box.
- **Notation:** connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$; quantifiers $\forall$, $\exists$; complement $\overline{A}$, difference $A-B$; numbers of selections and arrangements $\binom{n}{k}$, $P(n,k)$; congruence $a \equiv b \pmod{m}$; greatest common divisor and least common multiple gcd, lcm; graph $G=(V,E)$, degree $\deg(v)$. In truth tables, T means true and F false. In the figures, the path, circuit, or tree asked for is drawn thick and in color.
- **Proofs:** The solution of a proof problem first names the method (direct proof, proof by contraposition, proof by contradiction, induction, ...) and then proceeds in numbered steps. Try to write your own proofs in the same order.
- **Midterm and final:** In most schools the midterm and final topics are split as follows:

Midterm	Ch. 1–6	logic, methods of proof, sets and functions, algorithms, number theory, induction
Final	Ch. 7–13	counting, discrete probability, recurrence relations, relations, graphs, trees, Boolean algebra

Chapters marked *advanced* (8 and 13) are taught only in some schools. In courses that emphasize relations, lattices, Boolean algebra, and automata, Chapters 10 and 13 come to the fore. Choose the chapters to study according to your own course syllabus.

Contents

Page numbers refer to the full book. Chapters in color are in this sample; click to go there.

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4 Algorithms and the Growth of Functions ..... 10 problems 26
5 Number Theory and Cryptography ..... 14 problems 32
6 Induction and Recursion ..... 14 problems 38
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8 Discrete Probability (advanced) ..... 10 problems 53
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Solutions 98

CHAPTER 1
PROPOSITIONAL LOGIC

conditional	$p \to q \equiv \neg p \vee q \equiv \neg q \to \neg p$	distributive	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
biconditional	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$		$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
exclusive or	$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$	absorption	$p \vee (p \wedge q) \equiv p, \quad p \wedge (p \vee q) \equiv p$
De Morgan	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	negation	$p \vee \neg p \equiv \mathrm{T}, \quad p \wedge \neg p \equiv \mathrm{F}$
	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	identity, domination	$p \wedge \mathrm{T} \equiv p, \quad p \vee \mathrm{T} \equiv \mathrm{T}$

Unless stated otherwise, the following conventions hold. In truth tables, T means true and F means false; the rows are ordered so that T values come first. In equations, T also denotes the proposition that is always true, and F the proposition that is always false. A proposition that is true under every assignment is a tautology, one that is false under every assignment is a contradiction, and one that is neither is a contingency. Disjunctive normal form (DNF) consists of terms built with $\wedge$ and joined by $\vee$; conjunctive normal form (CNF) consists of terms built with $\vee$ and joined by $\wedge$. The precedence of the connectives is $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$, in that order; parentheses are still written out. The phrase "q only if p" means the one-way $q \to p$ (p is a necessary condition for q); "if and only if" is used for both directions.

Propositions, connectives, and truth tables

1.3 ★☆☆ Solution →

Which of the following propositions is a tautology?

A) $(p \to q) \to (q \to p)$

B) $(p \vee q) \to (p \wedge q)$

C) $(p \to q) \to (\neg p \to \neg q)$

D) $(p \to q) \vee (q \to p)$

E) $(p \leftrightarrow q) \to (p \wedge q)$

Logical equivalence

1.6 ★★☆ Solution →

Without using a truth table, prove the following equivalence using only the laws of logical equivalence:

$$(p \to q) \wedge (p \vee q) \wedge (q \to r) \equiv q \wedge r.$$

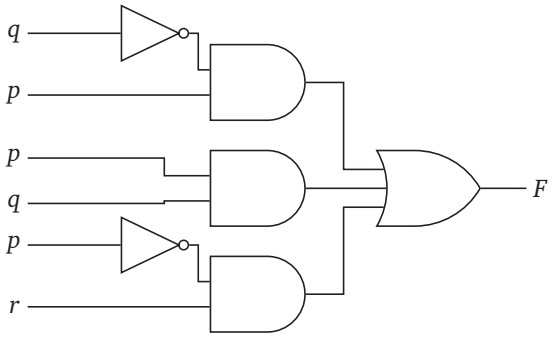
Normal forms and logic circuits

1.11 ★★☆ Solution →

The logic circuit in the figure has inputs p, q, r and output F. Wires labeled with the same letter are connected to the same input.

a. Write F as a proposition in terms of the inputs.

b. Simplify the proposition using the laws of equivalence. Propose a single-gate circuit that does the same job.



CHAPTER 7

COUNTING

sum rule ($A \cap B = \varnothing$) $|A \cup B| = |A| + |B|$

product rule $|A \times B| = |A| \cdot |B|$

subtraction rule $|A \cup B| = |A| + |B| - |A \cap B|$

pigeonhole principle

N objects, k boxes $\Rightarrow$ some box $\geq \lceil N/k \rceil$

permutation $P(n,k) = \dfrac{n!}{(n-k)!}$

combination $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!} = \binom{n}{n-k}$

Pascal's identity $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

binomial theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

permutations of a multiset $\dfrac{n!}{n_1!\,n_2! \cdots n_m!}$

combinations with repetition $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$

Unless stated otherwise, the following assumptions hold. People and objects are distinguishable from one another. "Choosing" means an unordered selection; "arranging" or "assigning roles" means an ordered selection. The digits are 0–9; in codes and passwords, position matters. $\lceil x \rceil$ is the smallest integer not less than x; if $k > n$, we take $\binom{n}{k} = 0$. In tree diagrams each branch represents a choice and each leaf an outcome.

The pigeonhole principle

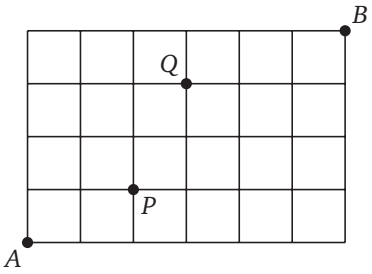
7.6 ★★☆ Solution →

13 points are placed inside or on the sides of a rectangle with side lengths 8 and 3 units. Show that some two of these points are at most $\sqrt{5}$ units apart.

Permutations and combinations

7.10 ★★☆ Solution →

In an exhibition ground with a grid plan, a visitor will walk from entrance A to exit B in the figure, moving only east (right) or north (up). Each line is a corridor, and each intersection is a junction.



a. How many different paths are there?

b. How many of them pass the information desk at junction P?

c. Junction Q is closed because a stand is being set up there. How many paths pass through P but not through Q?

Mixed problems

7.16 ★★★ Solution →

On a 14-day domestic tour, 5 days will be free days and the other 9 days will be guided sightseeing days. The tour company does not want two free days in a row.

a. In how many different ways can the free days be chosen?

b. Each of the five free days will get a different one of five optional activities (boat tour, museum, hike, cooking workshop, bike tour). How many different tour plans can be made?

c. If the first and last days of the tour cannot be free days either, what is the answer to part a?

CHAPTER 11

GRAPHS

handshaking theorem $\sum_{v\in V}\deg(v)=2|E|$

directed graph $\sum_v \deg^-(v)=\sum_v \deg^+(v)=|E|$

v_i–v_j paths of length k $(A^k)_{ij}$

Euler circuit connected, all degrees even

Euler path

connected, exactly two odd-degree vertices

Hamilton (Dirac)

$n\geq 3,\ \deg(v)\geq n/2 \Rightarrow$ circuit exists

Dijkstra $L(v)\leftarrow \min\{L(v),\ L(u)+w(u,v)\}$

Euler's formula $v-e+r=2$

planar, $v\geq 3$ $e\leq 3v-6$

Kuratowski planar $\Leftrightarrow$ no subdivision of K_5 or $K_{3,3}$

chromatic number

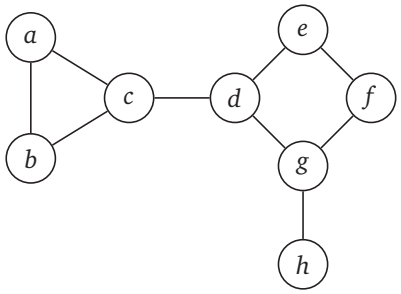
$\chi(G)\geq$ vertex count of the largest complete subgraph

Unless stated otherwise, the following assumptions hold. Graphs are finite, undirected, and simple (no loops and no multiple edges). Edges and vertices may repeat in a path; the length of a path is its number of edges. A path that ends at the vertex where it starts is called a circuit, and a circuit with distinct vertices and at least three edges is called a cycle. An Euler path (circuit) passes through every edge exactly once, and a Hamilton circuit passes through every vertex exactly once. In a planar drawing, v, e, and r are the numbers of vertices, edges, and regions. In the figures, the required path, circuit, or subgraph is drawn thick in the chapter color, and only in the solution.

Isomorphism and connectivity

11.5

★☆☆ Solution →



The fiber connections of eight buildings (a–h) on a campus are shown by the graph in the figure. An edge whose removal increases the number of connected components is called a *cut edge*, and a vertex whose removal, together with its edges, increases the number of components is called a *cut vertex*.

a. Find the cables that split the network when cut on their own (the cut edges).

b. Find the buildings that split the network when they alone fail (the cut vertices).

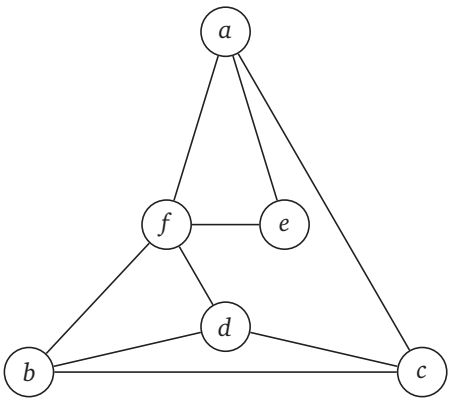
c. Between which two buildings can a single new cable be laid so that no cut edge remains? Write all the possibilities.

Planarity and coloring

11.13

★★★ Solution →

In a planar drawing of a connected planar simple graph ($v \geq 3$), suppose that every region, including the outer region, has exactly three edges on its boundary.



a. Prove that $e = 3v - 6$ and $r = 2v - 4$.

b. What is the maximum number of edges that can be added to the planar graph in the figure so that the graph stays simple and planar? Add that many edges and draw the graph without crossings.

c. A connected planar simple graph has 12 vertices, and every region is a triangle. If all vertices have the same degree, what is this degree?

SOLUTIONS

1 PROPOSITIONAL LOGIC

1.3

← Problem

For a conditional statement or a disjunction, we look for an assignment that makes it false. For the disjunction in (D) to be false, both conditional statements must be false. If $p \to q$ is false, then p is true and q is false; then $q \to p$ is true. The two cannot be false together; the proposition is true under every assignment. **D** is correct.

- (A): When p is false and q is true, $p \to q$ is true and $q \to p$ is false; the proposition is false. The converse of a conditional statement is not equivalent to it.
- (B): When p is true and q is false, $p \vee q$ is true and $p \wedge q$ is false.
- (C): When p is false and q is true, $p \to q$ is true and $\neg p \to \neg q$ is false. The inverse is not equivalent to the conditional statement either.
- (E): When p and q are both false, $p \leftrightarrow q$ is true and $p \wedge q$ is false.

ANSWER D

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1.6

← Problem

Method: direct proof (chain of equivalences).

1. Write the conditional statement as a disjunction: $p \to q \equiv \neg p \vee q$.
2. Rearrange the first two factors by the commutative law and apply the distributive law:
$$(q \vee \neg p) \wedge (q \vee p) \equiv q \vee (\neg p \wedge p).$$
3. By the negation law $\neg p \wedge p \equiv \mathrm{F}$, and by the identity law $q \vee \mathrm{F} \equiv q$. The first two factors are equivalent to q.
4. The remaining expression is $q \wedge (q \to r) \equiv q \wedge (\neg q \vee r)$. By the distributive law,
$$(q \wedge \neg q) \vee (q \wedge r) \equiv \mathrm{F} \vee (q \wedge r) \equiv q \wedge r.$$

Each step is a law of equivalence; the two sides are equivalent. ■

Check (one assignment): when p is true, q is true, and r is false, the left side is $\mathrm{T} \wedge \mathrm{T} \wedge \mathrm{F} = \mathrm{F}$ and the right side is $\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$ ✓.

Common mistake: In step 2, writing $(\neg p \vee q) \wedge (p \vee q)$ as $(\neg p \vee p) \wedge q$ instead of $(\neg p \wedge p) \vee q$; the distributive law factors out the common term (q).

ANSWER Proof above: the first two factors are equivalent to q, and $q \wedge (q \to r)$ is equivalent to $q \wedge r$.

1.11

← Problem

a. From top to bottom, the outputs of the three AND gates are $p \wedge \neg q$, $p \wedge q$, and $\neg p \wedge r$. The OR gate combines them:

$$F = (p \wedge \neg q) \vee (p \wedge q) \vee (\neg p \wedge r).$$

b. The first two terms have p in common (distributive, negation, and identity laws):

$$(p \wedge \neg q) \vee (p \wedge q) \equiv p \wedge (\neg q \vee q) \equiv p \wedge \mathrm{T} \equiv p.$$

In the remaining expression, use the distributive law of

$\vee$ over $\wedge$:

$$p \vee (\neg p \wedge r) \equiv (p \vee \neg p) \wedge (p \vee r) \equiv p \vee r.$$

Thus $F \equiv p \vee r$; the input q has no effect on the output. A single OR gate with inputs p and r does the same job.

Check: When p and r are false, the first two AND gates in the circuit output F because of p, and the third because of r; F is false. $p \vee r$ is also false ✓.

ANSWER a. $F = (p \wedge \neg q) \vee (p \wedge q) \vee (\neg p \wedge r)$
b. $F \equiv p \vee r$; a single OR gate

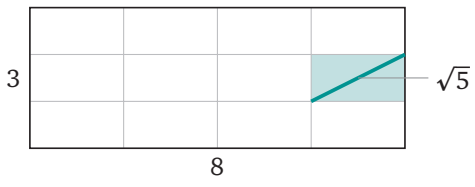
7 COUNTING

7.6

← Problem

Method: pigeonhole principle.

1. Divide the rectangle into $4 \cdot 3 = 12$ small rectangles with sides 2 and 1 unit (see figure). The small rectangles are the pigeonholes, and the points are the pigeons.
2. A point on the common side of two small rectangles is assigned to only one of them (for example, the left one or the lower one). Thus each point falls into exactly one pigeonhole.
3. Since 13 points are distributed among 12 pigeonholes, by the pigeonhole principle some pigeonhole contains at least $\lceil 13/12 \rceil = 2$ points.
4. The distance between two points of a rectangle with sides 2 and 1 unit (boundary included) cannot exceed its diagonal: $d \le \sqrt{2^2 + 1^2} = \sqrt{5}$.



The two points in this pigeonhole form the required pair. ■

ANSWER Proof above: 12 pigeonholes, 13 points; two points in one pigeonhole, distance $\le \sqrt{5}$.

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7.10

← Problem

a. Each path is a 10-letter string of 6 east (E) and 4 north (N) steps; a path is determined by which 4 of the 10 positions hold the N steps:

$$\binom{10}{4} = 210.$$

b. $A(0,0) \to P(2,1)$: 2 E, 1 N; $P \to B(6,4)$: 4 E, 3 N. By the product rule:

$$\binom{3}{1} \cdot \binom{7}{3} = 3 \cdot 35 = 105.$$

c. Paths through both P and $Q(3,3)$: $A \to P$: 3; $P \to Q$: 1 E, 2 N, $\binom{3}{1} = 3$; $Q \to B$: 3 E, 1 N, $\binom{4}{1} = 4$:

$$3 \cdot 3 \cdot 4 = 36, \qquad 105 - 36 = 69.$$

Check: The $A \to P$ paths can be counted by hand: EEN, ENE, NEE; 3 paths, $\binom{3}{1} = 3$ ✓. The $Q \to B$ paths are $EEEN$, $EENE$, $ENEE$, $NEEE$; 4 paths ✓.

ANSWER a. 210 b. 105 c. 69

7.16

← Problem

a. First line up the 9 guided days (G). There are 10 gaps: 8 between them and 2 at the two ends:

$$_\, G \,_\, G \,_\, G \,_\, G \,_\, G \,_\, G \,_\, G \,_\, G \,_$$

If at most one free day is placed in each gap, no two free days are consecutive; conversely, every selection with no two consecutive days is obtained in exactly one way like this. Choose 5 of the 10 gaps:

$$\binom{10}{5} = 252.$$

b. For each set of free days, the 5 different activities can be assigned to the 5 days in $5! = 120$ ways (product rule):

$$252 \cdot 120 = 30240.$$

c. The gaps at the two ends cannot be used; only the 8 gaps between the guided days remain:

$$\binom{8}{5} = 56.$$

Check (a small case): for 2 nonconsecutive free days in 5 days, the formula gives $\binom{4}{2} = 6$. Direct list: $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{3,5\}$; 6 sets ✓.

Common mistake: writing $\binom{14}{5} = 2002$ in part a; this also counts consecutive free days.

ANSWER a. 252 b. 30240 c. 56

11 GRAPHS

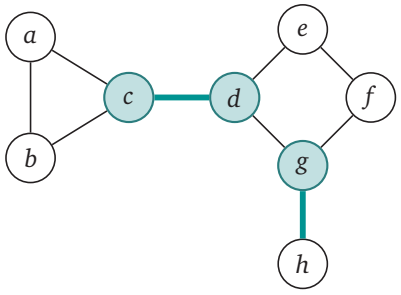
11.5

← Problem

a. An edge is a cut edge only if it lies on no cycle: when an edge on a cycle is removed, its endpoints remain connected through the rest of the cycle. The edges of the triangle a, b, c and of the cycle d, e, f, g are not cut edges. The remaining two edges are $\{c,d\}$ and $\{g,h\}$; removing $\{c,d\}$ separates $\{a,b,c\}$ from $\{d,e,f,g,h\}$, and removing $\{g,h\}$ separates h.

b. Removing c separates $\{a,b\}$ from $\{d,\ldots,h\}$; removing d separates $\{a,b,c\}$ from $\{e,f,g,h\}$; removing g separates h from the others. When any one of a, b, e, f, h is removed, the remaining graph is connected.

c. The new cable must create a cycle that contains both cut edges. The old path between the endpoints of the cable must pass through $\{c,d\}$ and $\{g,h\}$; this happens when one endpoint is h and the other is a, b, or c. For example, adding $\{h,c\}$ creates the cycle h,g,d,c,h.



ANSWER a. $\{c,d\}$ and $\{g,h\}$ b. c, d, g c. $\{h,a\}$, $\{h,b\}$, or $\{h,c\}$

11.13

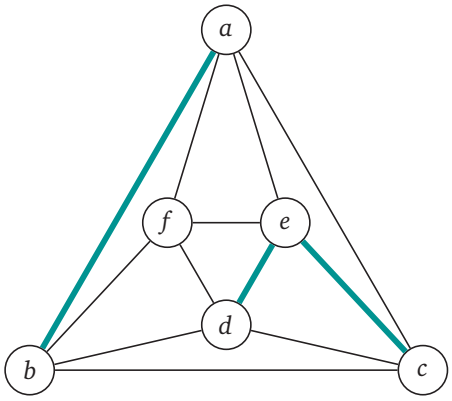
← Problem

a. Method: direct proof (counting region boundaries in two ways).

1. Add up the numbers of edges on the boundaries of all regions. Each edge is counted twice (once for the region on each side); the total is $2e$.
2. Since each region has 3 edges on its boundary, the total is $3r$: $3r = 2e$, that is, $r = 2e/3$.
3. Substituting into Euler's formula gives $v - e + 2e/3 = 2$; hence $e = 3v - 6$.
4. $r = 2e/3 = 2(3v-6)/3 = 2v - 4$.

Both equalities are proved. ■

b. The graph has $v = 6$ and $e = 9$. Since $e \le 3v - 6 = 12$ in a planar simple graph, at most 3 edges can be added. Three edges can indeed be added: a–b, d–e, and c–e (thick in the figure) create no crossings.



In the new graph $e = 12 = 3v - 6$, and all regions are triangles; there are $r = 2 \cdot 6 - 4 = 8$ regions.

c. By part a, $e = 3 \cdot 12 - 6 = 30$. The degree sum is $2e = 60$; since the twelve vertices have equal degrees, each degree is $60/12 = 5$. (Example: the graph whose vertices and edges are the vertices and edges of a regular icosahedron.)

ANSWER a. proof above b. at most 3 edges (a–b, d–e, c–e) c. 5

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