

DIFFERENTIAL EQUATIONS

WORKBOOK

13 chapters · 172 problems · step-by-step solutions



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CHAPTER 1

BASIC CONCEPTS

order	highest derivative in the equation	uniqueness: linear	$y' + p(x)y = g(x), x_0 \in I$
linear equation	$a_n(x)y^{(n)} + \dots + a_0(x)y = g(x)$	p, g continuous on $I \implies$	unique solution on I
initial value problem	$y' = f(x, y), y(x_0) = y_0$	uniqueness: nonlinear	$y' = f(x, y), y(x_0) = y_0$
Euler's method	$y_{n+1} = y_n + hf(x_n, y_n), x_{n+1} = x_n + h$	$f, \partial f / \partial y$ continuous $\implies$	unique solution near x_0

Unless stated otherwise, the independent variable is x , the unknown function is $y = y(x)$, and $y' = dy/dx$. Questions on existence and uniqueness use the two theorems above; for a nonlinear equation the theorem does not say how large the interval is.

Classification and the notion of a solution

1.1

☆☆☆ Solution → p. 100

Which of the following is a second-order linear ordinary differential equation?

- A) $y'' + y y' = \cos x$
- B) $y'' + \sin y = x$
- C) $y''' - x y = 0$
- D) $u_{xx} + u_{yy} = 0$
- E) $x^2 y'' - e^x y' + y = \ln x$

1.2

☆☆☆ Solution → p. 100

Give the order of each equation and state whether it is ordinary or partial, linear or nonlinear. For each nonlinear equation, identify the term that makes it nonlinear.

- a. $x^2 y''' + e^x y' = y + \cos x$
- b. $y' + x\sqrt{y} = 0$
- c. $u_{tt} = 4u_{xx} + u$ ($u = u(x, t)$)
- d. $(y'')^2 + 3y' = x$

1.3

★★★ Solution → p. 101

- Show that $y = Cx e^{-x^2/2}$ is a solution of $x y' = (1 - x^2) y$ for every constant C .
- Find the member of this family that satisfies $y(1) = 2$.
- For which values of r is $y = e^{rx}$ a solution of $2y'' + y' - 3y = 0$?

1.4

★★★ Solution → p. 101

Consider the relation $y^2 - 2xy + 3x^2 = C$.

- Show that this relation is an implicit solution of $y' = (y - 3x)/(y - x)$.
- Write the solution through the point $(1, 3)$ first in implicit form, then in explicit form.
- What is the largest interval on which the explicit solution is defined? What happens at the endpoints of the interval?

Initial value problems and modeling

1.5

★★★☆☆ Solution → p. 101

Consider the family $y = C_1 e^x + C_2 e^{-4x} + 2$.

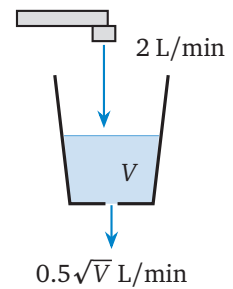
- Show that every member of the family is a solution of $y'' + 3y' - 4y = -8$.
- Solve the initial value problem with $y(0) = 5$, $y'(0) = -2$.
- Under what condition does the solution satisfying $y(0) = a$, $y'(0) = b$ remain bounded as $x \rightarrow \infty$? What does the solution approach in that case?

1.6

★★★☆☆ Solution → p. 102

Water flows from a tap into an empty bucket at 2 L per minute. When the bucket holds V liters, water leaks out through a small hole in the bottom at $0.5\sqrt{V}$ liters per minute (V in liters). The capacity of the bucket is 20 L.

- Set up the differential equation for the volume $V(t)$ of water in the bucket (t in minutes). What is its order? Is it linear?
- Find the equilibrium solution. Does the bucket overflow if the tap is left open?
- If the bucket starts full, with the same tap and hole, how does the volume change? Explain using the direction field.



SOLUTIONS

1 BASIC CONCEPTS

1.1

← Problem p. 4

In a linear equation, y and its derivatives appear only to the first power, are not multiplied together and do not appear inside a function (sine, square root...); the coefficients may depend only on x . In (E) the highest derivative is y'' (order 2), the coefficients x^2 , $-e^x$, 1 depend only on x , and the right-hand side is $\ln x$: it is a second-order linear ordinary differential equation.

- (A): The product $y y'$ makes it nonlinear.
- (B): The term $\sin y$ is a nonlinear function of y .
- (C): It is linear but of third order.
- (D): It is a partial differential equation (u is a function of two variables).

ANSWER E

1.2

← Problem p. 4

a. The highest derivative is y''' : order 3. There is a single independent variable: ordinary. It can be written as $x^2 y''' + e^x y' - y = \cos x$; y and its derivatives appear to the first power and the coefficients depend only on x : linear.

b. Order 1, ordinary. The term $\sqrt{y}$ is a nonlinear function of y : nonlinear.

c. $u = u(x, t)$ depends on two variables: partial. The highest derivatives are of second order: order 2. u and its partial derivatives appear to the first power: linear.

d. The highest derivative is y'' : order 2 (the power of the derivative, here 2, does not change the order). Ordinary. Nonlinear because of the term $(y'')^2$.

Common mistake: Seeing the term $(y'')^2$ and taking the order to be 4. The order is the order of the derivative, not its power.

ANSWER a. 3, ordinary, linear b. 1, ordinary, nonlinear ($\sqrt{y}$) c. 2, partial, linear d. 2, ordinary, nonlinear $((y'')^2)$

1.3

← Problem p. 5

a. By the product rule,

$$y' = C e^{-x^2/2} + Cx(-x)e^{-x^2/2} = C(1-x^2)e^{-x^2/2}.$$

Hence

$$x y' = (1-x^2) Cx e^{-x^2/2} = (1-x^2) y \checkmark$$

The equality holds for every x and every C .b. $y(1) = C e^{-1/2} = 2$ gives $C = 2e^{1/2}$:

$$y = 2x e^{(1-x^2)/2}.$$

c. $y' = r e^{rx}$, $y'' = r^2 e^{rx}$; substituting these into the equation gives

$$(2r^2 + r - 3) e^{rx} = 0.$$

Since $e^{rx} \neq 0$, $2r^2 + r - 3 = (r-1)(2r+3) = 0$:

$$r = 1 \quad \text{or} \quad r = -\frac{3}{2}.$$

ANSWER b. $y = 2x e^{(1-x^2)/2}$ c. $r = 1$, $r = -\frac{3}{2}$

1.4

← Problem p. 5

a. Regard y as $y(x)$ and differentiate both sides of the relation with respect to x (implicit differentiation):

$$2y y' - 2y - 2x y' + 6x = 0.$$

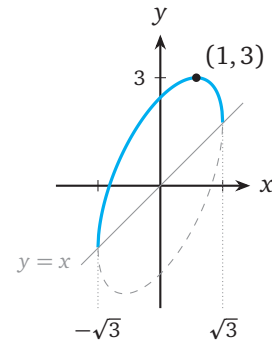
$$y'(2y - 2x) = 2y - 6x \implies y' = \frac{y - 3x}{y - x} \checkmark$$

b. $(1, 3)$ gives $C = 9 - 6 + 3 = 6$; the implicit solution is $y^2 - 2xy + 3x^2 = 6$. This is a quadratic equation in y :

$$y = x \pm \sqrt{x^2 - (3x^2 - 6)} = x \pm \sqrt{6 - 2x^2}.$$

Since $1 + \sqrt{4} = 3$ at $x = 1$, the plus sign is taken:

$$y = x + \sqrt{6 - 2x^2}.$$

c. The expression under the square root must be positive: $6 - 2x^2 > 0$, that is, $-\sqrt{3} < x < \sqrt{3}$. At the endpoints $y = x$; the denominator $y - x$ of the equation vanishes and the tangent to the solution curve is vertical. The implicit solution is an ellipse; the explicit solution is its upper half (figure).The dashed lower half ($y = x - \sqrt{6 - 2x^2}$) is the other solution given by the same relation; it does not pass through $(1, 3)$.**ANSWER** b. $y^2 - 2xy + 3x^2 = 6$; $y = x + \sqrt{6 - 2x^2}$
c. $-\sqrt{3} < x < \sqrt{3}$; vertical tangent at the endpoints

1.5

← Problem p. 6

a. $y' = C_1 e^x - 4C_2 e^{-4x}$, $y'' = C_1 e^x + 16C_2 e^{-4x}$.

Substituting,

$$\begin{aligned} y'' + 3y' - 4y &= (1 + 3 - 4) C_1 e^x \\ &\quad + (16 - 12 - 4) C_2 e^{-4x} - 8 \\ &= -8 \checkmark \end{aligned}$$

b. The initial conditions give

$$C_1 + C_2 + 2 = 5, \quad C_1 - 4C_2 = -2.$$

Subtracting gives $5C_2 = 5$; $C_2 = 1$, $C_1 = 2$:

$$y = 2e^x + e^{-4x} + 2.$$

c. Since $e^{-4x} \rightarrow 0$, the solution remains bounded only if the coefficient of the e^x term is zero. From the conditions,

$$C_1 + C_2 = a - 2, \quad C_1 - 4C_2 = b.$$

Multiplying the first by 4 and adding the second gives $5C_1 = 4a + b - 8$. For $C_1 = 0$,

$$b + 4a = 8.$$

In this case $y = C_2 e^{-4x} + 2 \rightarrow 2$. The constant $y = 2$ is also a solution ($C_1 = C_2 = 0$).**ANSWER** b. $y = 2e^x + e^{-4x} + 2$ c. $b + 4a = 8$;
 $y \rightarrow 2$