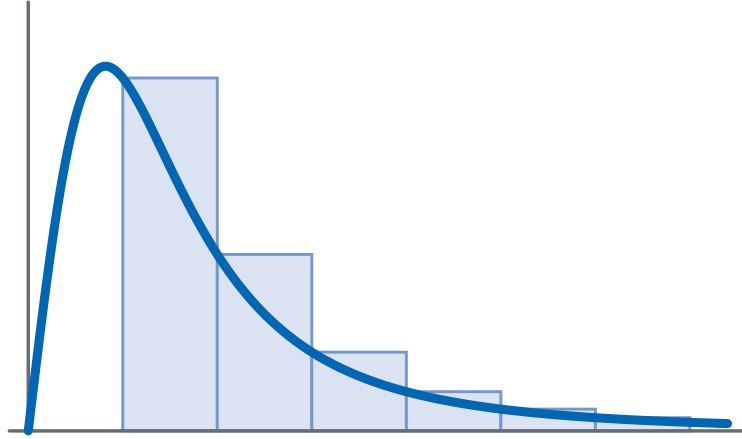


# CALCULUS 1

## WORKBOOK

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9 chapters · 154 problems · step-by-step solutions



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# CHAPTER 2

## DERIVATIVES



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|            |                                                          |                              |                                         |
|------------|----------------------------------------------------------|------------------------------|-----------------------------------------|
| definition | $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ | inverse function, $b = f(a)$ | $(f^{-1})'(b) = \frac{1}{f'(a)}$        |
| product    | $(uv)' = u'v + uv'$                                      | inverse sine                 | $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ |
| quotient   | $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$      | inverse tangent              | $(\arctan x)' = \frac{1}{1+x^2}$        |
| chain rule | $(f(g(x)))' = f'(g(x))g'(x)$                             | linearization                | $L(x) = f(a) + f'(a)(x-a)$              |

### The definition of the derivative

2.1

★★★ Solution → p. 91

The function  $f(x) = \frac{2}{1 + \sqrt{x}}$  is given.

- Find  $f'(9)$  using the limit definition of the derivative.
- Find the point where the tangent line to the curve at  $x = 9$  meets the  $x$ -axis.

Grid area for working on problem 2.1.

2.2

★★★ Solution → p. 91

Which of the following functions is differentiable at  $x = 0$ ?

- A)  $|x|$       B)  $\sqrt[3]{x}$       C)  $x|x|$       D)  $|\sin x|$       E)  $x^{2/3}$

Grid area for working on problem 2.2.

## Differentiation rules

2.3

★★★ Solution → p. 91

Find the points on the curve  $f(x) = \frac{x^2 + 3}{x + 1}$  where the tangent is horizontal.

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2.4

★★★ Solution → p. 91

The graph of  $f(x) = (ax + b)e^x$  passes through the point  $(0, 1)$ , and the slope of the tangent at this point is 3.

- Find the constants  $a$  and  $b$ .
- Compute  $f''(0)$ .

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2.5

★★★ Solution → p. 91

An object moves along a line so that its position at time  $t$  seconds is  $s(t) = t^3 - 15t^2 + 63t$  meters ( $0 \leq t \leq 8$ ).

- Find the velocity and acceleration of the object. At what times does the object change direction?
- Find the total distance traveled by the object in the first 8 seconds and its average velocity.
- Find the speed at the moment the acceleration is zero. On which time intervals is the object slowing down?

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**2.6**

★★★ Solution → p. 92

The revenue obtained when  $x$  tickets are sold in a concert hall is  $R(x) = 120x - 0.4x^2$  TL ( $0 \leq x \leq 300$ ).

- Find the marginal revenue  $R'(x)$  and compute  $R'(100)$ .
- Compute the actual additional revenue brought by the 101st ticket and compare it with  $R'(100)$ .
- At what sales level does the revenue stop increasing?

Grid area for problem 2.6.

**2.7**

★★★ Solution → p. 92

Find the points on the curve  $y = \frac{\sin x}{2 + \cos x}$  in the interval  $[0, 2\pi]$  where the tangent is horizontal.

Grid area for problem 2.7.

## 2 DERIVATIVES

### 2.1

← Problem p. 12

a.  $f(9) = \frac{2}{1+3} = \frac{1}{2}$ . For brevity let  $s = \sqrt{9+h}$ . Let us put the difference quotient over a common denominator:

$$\begin{aligned}\frac{f(9+h) - f(9)}{h} &= \frac{1}{h} \left( \frac{2}{1+s} - \frac{1}{2} \right) \\ &= \frac{3-s}{2h(1+s)}.\end{aligned}$$

As  $h \rightarrow 0$ ,  $s \rightarrow 3$ ; the numerator and denominator both tend to zero (0/0). Let us rationalize the numerator:  $(3-s)(3+s) = 9 - (9+h) = -h$ . Thus

$$\begin{aligned}\frac{f(9+h) - f(9)}{h} &= \frac{-h}{2h(1+s)(3+s)} \\ &= \frac{-1}{2(1+s)(3+s)}\end{aligned}$$

and after cancelling we may put  $h = 0$  ( $s = 3$ ):

$$f'(9) = \frac{-1}{2 \cdot 4 \cdot 6} = -\frac{1}{48}.$$

**Check:** By the Chain Rule,  $f'(x) = -\frac{2}{(1+\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} = -\frac{1}{\sqrt{x}(1+\sqrt{x})^2}$ ;  $f'(9) = -\frac{1}{3 \cdot 16} = -\frac{1}{48}$ .

b. The tangent passes through  $(9, \frac{1}{2})$  and has slope  $-\frac{1}{48}$ :

$$y - \frac{1}{2} = -\frac{1}{48}(x - 9) \implies y = -\frac{x}{48} + \frac{11}{16}.$$

Setting  $y = 0$  gives  $\frac{x}{48} = \frac{11}{16}$ , that is,  $x = 33$ .

**ANSWER** a.  $-\frac{1}{48}$  b.  $(33, 0)$  (tangent:  $y = -\frac{x}{48} + \frac{11}{16}$ )

### 2.2

← Problem p. 12

For each function let us examine the difference quotient at  $x = 0$ , that is,  $\frac{f(h) - f(0)}{h} = \frac{f(h)}{h}$ .

- $|x|$ :  $\frac{|h|}{h}$  tends to 1 from the right and to  $-1$  from the left. The one-sided derivatives differ: a corner.
- $\sqrt[3]{x}$ :  $\frac{h^{1/3}}{h} = h^{-2/3} \rightarrow +\infty$ . There is a vertical tangent, no derivative.
- $x|x|$ :  $\frac{h|h|}{h} = |h| \rightarrow 0$ . The limit is 0 from both sides;  $f'(0) = 0$ .
- $|\sin x|$ :  $\frac{|\sin h|}{h}$  tends to 1 from the right and to  $-1$  from the left: a corner.

- $x^{2/3}$ :  $\frac{h^{2/3}}{h} = h^{-1/3}$  tends to  $+\infty$  from the right and to  $-\infty$  from the left: a cusp.

The function  $x|x|$  equals  $x^2$  for  $x \geq 0$  and  $-x^2$  for  $x < 0$ . The two pieces join at  $x = 0$  with the same slope (zero); the absolute value does not create a corner here.

**ANSWER** C

### 2.3

← Problem p. 13

By the Quotient Rule:

$$\begin{aligned}f'(x) &= \frac{2x(x+1) - (x^2+3) \cdot 1}{(x+1)^2} \\ &= \frac{x^2+2x-3}{(x+1)^2} = \frac{(x-1)(x+3)}{(x+1)^2}.\end{aligned}$$

The tangent is horizontal when  $f'(x) = 0$ . Since the denominator is nonzero for  $x \neq -1$ , only the numerator can vanish:  $x = 1$  or  $x = -3$ . At these points  $f(1) = \frac{4}{2} = 2$  and  $f(-3) = \frac{12}{-2} = -6$ .

**ANSWER**  $(1, 2)$  and  $(-3, -6)$

### 2.4

← Problem p. 13

a. Since  $f(0) = b \cdot e^0 = b$ , we have  $b = 1$ . By the Product Rule,

$$f'(x) = a e^x + (ax + b) e^x = (ax + a + b) e^x,$$

that is,  $f'(0) = a + b$ . From the condition  $f'(0) = 3$  we get  $a + 1 = 3$ , that is,  $a = 2$ .

b. Let us differentiate  $f'(x) = (2x + 3) e^x$  again with the Product Rule:

$$f''(x) = 2e^x + (2x + 3) e^x = (2x + 5) e^x,$$

that is,  $f''(0) = 5$ .

**ANSWER** a.  $a = 2$ ,  $b = 1$  b. 5

### 2.5

← Problem p. 13

a. Differentiating,

$$v(t) = s'(t) = 3t^2 - 30t + 63 = 3(t-3)(t-7),$$

$$a(t) = v'(t) = 6t - 30.$$

$v$  is zero at  $t = 3$  s and  $t = 7$  s and changes sign at these times (simple roots); the object changes direction at these two instants.

b. For the total distance the interval must be split at the times the direction changes. Since  $s(0) = 0$ ,  $s(3) = 81$ ,  $s(7) = 49$ ,  $s(8) = 56$ :

$$|81 - 0| + |49 - 81| + |56 - 49| \\ = 81 + 32 + 7 = 120 \text{ m.}$$

The average velocity, however, is the displacement divided by the elapsed time:  $\frac{s(8) - s(0)}{8} = \frac{56}{8} = 7 \text{ m/s}$ . Common mistake: dividing the total distance (120 m) by the time; this gives the average speed (15 m/s), not the average velocity.

c.  $a = 0 \Rightarrow t = 5 \text{ s}$ . At this instant  $v(5) = 3(2)(-2) = -12 \text{ m/s}$ , so the speed is  $|v| = 12 \text{ m/s}$ . The speed decreases when  $v$  and  $a$  have opposite signs. The signs:

|     | (0, 3) | (3, 5) | (5, 7) | (7, 8) |
|-----|--------|--------|--------|--------|
| $v$ | +      | -      | -      | +      |
| $a$ | -      | -      | +      | +      |

The object slows down on the intervals (0, 3) and (5, 7) and speeds up on the other two.

**ANSWER** a.  $v = 3(t-3)(t-7)$ ,  $a = 6t - 30$ ;  $t = 3$  and  $7 \text{ s}$  b. 120 m; 7 m/s c. 12 m/s; (0, 3) and (5, 7)

## 2.6

← Problem p. 14

a.  $R'(x) = 120 - 0.8x$ ,  $R'(100) = 120 - 80 = 40$  TL/ticket. When 100 tickets have been sold, selling one more increases the revenue by about 40 TL.

b. The actual additional revenue is

$$R(101) - R(100) = 8039.6 - 8000 \\ = 39.6 \text{ TL}$$

$R(101) = 12120 - 4080.4$ ,  $R(100) = 12000 - 4000$   
The marginal revenue (40 TL) is very close to this value: the derivative is a good approximation of the change caused by a one-unit increase.

c.  $R'(x) = 0 \Rightarrow x = 150$ . For  $x < 150$ ,  $R' > 0$  (revenue increases); for  $x > 150$ ,  $R' < 0$  (revenue decreases).

**ANSWER** a.  $120 - 0.8x$ ; 40 TL b. 39.6 TL  
c.  $x = 150$

## 2.7

← Problem p. 14

By the Quotient Rule, using  $(\sin x)' = \cos x$  and  $(2 + \cos x)' = -\sin x$ :

$$y' = \frac{\cos x (2 + \cos x) - \sin x (-\sin x)}{(2 + \cos x)^2} \\ = \frac{2 \cos x + \cos^2 x + \sin^2 x}{(2 + \cos x)^2} = \frac{2 \cos x + 1}{(2 + \cos x)^2}.$$

The denominator is never zero ( $\cos x \geq -1$ ). For  $y' = 0$  we need  $\cos x = -\frac{1}{2}$ :  $x = \frac{2\pi}{3}$  or  $x = \frac{4\pi}{3}$ .

$$y\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}/2}{2 - 1/2} = \frac{\sqrt{3}}{3}, \\ y\left(\frac{4\pi}{3}\right) = \frac{-\sqrt{3}/2}{3/2} = -\frac{\sqrt{3}}{3}.$$

**ANSWER**  $\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{3}\right)$  and  $\left(\frac{4\pi}{3}, -\frac{\sqrt{3}}{3}\right)$

## 2.8

← Problem p. 15

a. There are three layers from the outside in: the fourth power, cosine,  $3x$ .

$$y' = 4 \cos^3(3x) \cdot (-\sin(3x)) \cdot 3 = -12 \cos^3(3x) \sin(3x).$$

b. Quotient Rule; the derivative of the denominator by the Chain Rule:  $(\sqrt{x^2 + 1})' = \frac{x}{\sqrt{x^2 + 1}}$ .

$$y' = \frac{2\sqrt{x^2 + 1} - (2x + 1) \frac{x}{\sqrt{x^2 + 1}}}{x^2 + 1} \\ = \frac{2(x^2 + 1) - x(2x + 1)}{(x^2 + 1)^{3/2}} = \frac{2 - x}{(x^2 + 1)^{3/2}}.$$

c. The exponential function reproduces itself, multiplied by the derivative of the exponent:

$$y' = e^{\sin 2x} \cdot \cos 2x \cdot 2 = 2 \cos 2x e^{\sin 2x}.$$

**ANSWER** a.  $-12 \cos^3(3x) \sin(3x)$  b.  $\frac{2 - x}{(x^2 + 1)^{3/2}}$   
c.  $2 \cos 2x e^{\sin 2x}$