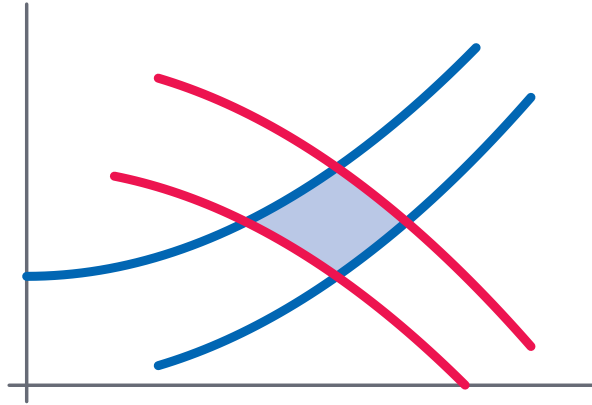


CALCULUS 2

WORKBOOK

10 chapters · 162 problems · step-by-step solutions



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CHAPTER 5

APPLICATIONS OF INTEGRATION



watch video

slicing

$$V = \int_a^b A(x) dx$$

surface area of revolution

$$S = 2\pi \int (\text{radius}) ds$$

washer method

$$V = \pi \int_a^b (R^2 - r^2) dx$$

work

$$W = \int_a^b F(x) dx$$

shell method

$$V = 2\pi \int_a^b (\text{radius})(\text{height}) dx$$

center of mass

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

arc length

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Pappus

$$V = 2\pi \rho A, \quad S = 2\pi \rho L$$

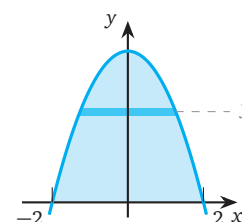
Slicing: Disk and Washer Methods

5.1

★★★ Solution → p. 93

The base of a solid is the region in the xy -plane between the parabola $y = 4 - x^2$ and the x -axis. Every cross section of the solid perpendicular to the y -axis has as its base a horizontal line segment lying in this region. The cross sections are

- squares,
- semicircles with this segment as diameter.



Find the volume of the solid in each case.

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5.2

★★★ Solution → p. 93

The region between the curve $y = \sec x$ and the line $y = 1$ for $0 \leq x \leq \frac{\pi}{3}$ is revolved about the x -axis. Find the volume of the resulting solid.

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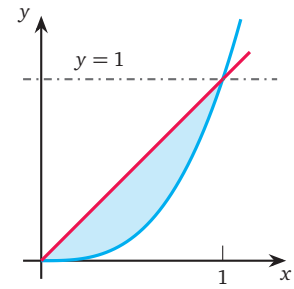
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5.3

★★★ Solution → p. 93

In the first quadrant, consider the region between the line $y = x$ and the curve $y = x^3$.

- The region is revolved about the line $y = 1$. Find the volume of the resulting solid.
- The region is revolved about the y -axis. Find the volume using the washer method.



Cylindrical Shells

5.4

★★★ Solution → p. 93

The region bounded by the curve $y = \frac{1}{\sqrt{25-x^2}}$, the x -axis, the y -axis, and the line $x = 3$ is revolved about the y -axis.

- Find the volume of the resulting solid by the shell method.
- Verify the result by computing the same volume with the disk and washer methods in y . Explain why this route is more laborious.

5.5

★★★ Solution → p. 94

The region R is bounded by the curve $y = 1/x$, the line $y = 1$, and the line $x = 3$. It is revolved about the line $x = 4$. Which of the following is the volume of the resulting solid?

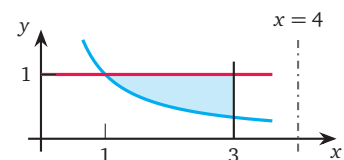
A) $2\pi \int_1^3 (x+4)\left(1-\frac{1}{x}\right) dx$

$$\text{B) } 2\pi \int_1^3 \frac{4-x}{x} dx$$

C) $2\pi \int_1^3 (4-x)\left(1-\frac{1}{x}\right) dx$

$$\text{D) } \pi \int_{1/3}^1 \left[1 - \left(4 - \frac{1}{y} \right)^2 \right] dy$$

E) $\pi \int_1^3 \left(1 - \frac{1}{x}\right)^2 dx$



SOLUTIONS

5 APPLICATIONS OF INTEGRATION

5.1

← Problem p. 4

Since the cross sections are perpendicular to the y -axis, we integrate with respect to y . At height y the endpoints of the horizontal segment satisfy $4 - x^2 = y$, so $x = \pm\sqrt{4-y}$. The length of the base of the cross section is $s(y) = 2\sqrt{4-y}$, $0 \leq y \leq 4$.

a. The area of a square cross section is $A(y) = s^2 = 4(4-y)$:

$$V = \int_0^4 4(4-y) dy = 4 \left[4y - \frac{y^2}{2} \right]_0^4 = 32.$$

b. The radius of the semicircle is $\frac{s}{2} = \sqrt{4-y}$, so its area is $A(y) = \frac{\pi}{2}(4-y)$:

$$V = \frac{\pi}{2} \int_0^4 (4-y) dy = \frac{\pi}{2} \cdot 8 = 4\pi.$$

Check: At every height the semicircle's area is $\frac{\pi}{8}$ of the square's area, so the volumes must be in the same ratio: $32 \cdot \frac{\pi}{8} = 4\pi$.

ANSWER a. 32 b. 4π

5.2

← Problem p. 4

The region does not touch the axis, so cross sections perpendicular to the x -axis are washers (disks with a hole). For $0 \leq x \leq \frac{\pi}{3}$ we have $\sec x \geq 1$, so the outer radius is $R = \sec x$ and the inner radius is $r = 1$:

$$\begin{aligned} V &= \pi \int_0^{\pi/3} (\sec^2 x - 1) dx = \pi [\tan x - x]_0^{\pi/3} \\ &= \pi \left(\sqrt{3} - \frac{\pi}{3} \right) \approx 2.15. \end{aligned}$$

Common mistake: writing $(R-r)^2 = (\sec x - 1)^2$ in the integrand. The area of a washer is $\pi R^2 - \pi r^2$, not $\pi(R-r)^2$.

ANSWER $\pi(\sqrt{3} - \frac{\pi}{3}) \approx 2.15$

5.3

← Problem p. 5

For $0 \leq x \leq 1$ we have $x^3 \leq x$; the line forms the upper boundary of the region and the curve the lower boundary.

a. The axis $y = 1$ lies above the region. The cross section at x is a washer. The boundary farther from the axis is the lower curve: $R = 1 - x^3$, $r = 1 - x$.

$$\begin{aligned} V &= \pi \int_0^1 [(1-x^3)^2 - (1-x)^2] dx \\ &= \pi \int_0^1 (2x - x^2 - 2x^3 + x^6) dx \\ &= \pi \left(1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{7} \right) = \frac{13\pi}{42}. \end{aligned}$$

b. Take cross sections perpendicular to the y -axis and write the boundaries as functions of y . At height y the region extends from $x = y$ to $x = y^{1/3}$: $R = y^{1/3}$, $r = y$.

$$V = \pi \int_0^1 (y^{2/3} - y^2) dy = \pi \left(\frac{3}{5} - \frac{1}{3} \right) = \frac{4\pi}{15}.$$

Check: By the shell method, $2\pi \int_0^1 x(x - x^3) dx = 2\pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{4\pi}{15}$.

ANSWER a. $\frac{13\pi}{42}$ b. $\frac{4\pi}{15}$

5.4

← Problem p. 6

a. Strips parallel to the axis (vertical) generate cylindrical shells: radius x , height $\frac{1}{\sqrt{25-x^2}}$.

$$V = 2\pi \int_0^3 \frac{x dx}{\sqrt{25-x^2}}.$$

With $u = 25 - x^2$, $du = -2x dx$, the limits become 25 and 16:

$$V = \pi \int_{16}^{25} u^{-1/2} du = \pi [2\sqrt{u}]_{16}^{25} = 2\pi(5-4) = 2\pi.$$

b. The curve rises from $y = \frac{1}{5}$ at $x = 0$ to $y = \frac{1}{4}$ at $x = 3$. There are two kinds of horizontal slices:

- $0 \leq y \leq \frac{1}{5}$: the slice extends from the y -axis to $x = 3$; the disk radius is 3.

- $\frac{1}{5} \leq y \leq \frac{1}{4}$: the slice extends from the curve to $x = 3$; for the inner radius of the washer, $x^2 = 25 - \frac{1}{y^2}$.

$$\begin{aligned} V &= \pi \cdot 9 \cdot \frac{1}{5} + \pi \int_{1/5}^{1/4} \left(9 - 25 + \frac{1}{y^2} \right) dy \\ &= \frac{9\pi}{5} + \pi \left[-16y - \frac{1}{y} \right]_{1/5}^{1/4} \\ &= \frac{9\pi}{5} + \pi \left(-8 + \frac{41}{5} \right) = \frac{9\pi}{5} + \frac{\pi}{5} = 2\pi. \end{aligned}$$

On this route we must write the curve as $x = g(y)$, split the region in two, and use washers on the second piece. With shells, a single integral suffices.

ANSWER a. 2π b. 2π (two-part integral)

5.5

← Problem p. 6

For $1 \leq x \leq 3$ we have $\frac{1}{x} \leq 1$; the line forms the upper boundary of the region and the curve the lower boundary. Since the axis is vertical, vertical strips generate shells. The strip at x has height $1 - \frac{1}{x}$ and distance $4 - x$ from the axis (the axis is to the right of the region):

$$V = 2\pi \int_1^3 (4-x) \left(1 - \frac{1}{x} \right) dx.$$

- (A) uses the radius $x + 4$; that is for the axis $x = -4$.
- (B) uses the height $\frac{1}{x}$; that revolves the region under the curve.
- (D) is the washer method, but the radii are mixed up. At height y the region R extends from $x = \frac{1}{y}$ to $x = 3$. The outer radius (farther from the axis) is $R = 4 - \frac{1}{y}$ and the inner radius is $r = 4 - 3 = 1$. The integrand of (D) is negative.
- (E) is the disk volume about the line $y = 1$.

Let us evaluate the integral in (C):

$$\begin{aligned} V &= 2\pi \int_1^3 \left(5 - x - \frac{4}{x} \right) dx \\ &= 2\pi (10 - 4 - 4 \ln 3) = 4\pi (3 - 2 \ln 3) \approx 10.09. \end{aligned}$$

Check: The correct washer integral gives the same value:

$$\begin{aligned} V &= \pi \int_{1/3}^1 \left[\left(4 - \frac{1}{y} \right)^2 - 1 \right] dy \\ &= \pi \int_{1/3}^1 \left(15 - \frac{8}{y} + \frac{1}{y^2} \right) dy \\ &= \pi (10 - 8 \ln 3 + 2) = 4\pi (3 - 2 \ln 3). \end{aligned}$$

ANSWER C