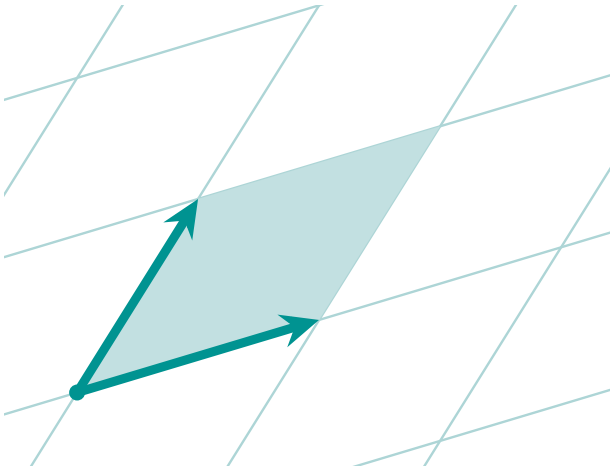
LINEAR ALGEBRA

WORKBOOK

10 chapters · 150 problems · step-by-step solutions



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CHAPTER 1

SYSTEMS OF LINEAR EQUATIONS

linear equation	$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$	number of solutions	0, 1 or ∞
row operations	$R_i \leftrightarrow R_j$, $R_i \to cR_i$ $(c \neq 0)$, $R_i \to R_i + cR_j$	free variable	number of unknowns − number of pivots
inconsistency	$\left[\begin{array}{ccc\|c} 0 & 0 & \cdots & 0 \end{array}\right]$, $c \neq 0$	homogeneous system	number of unknowns > number of equations ⇒ infinitely many solutions

Linear equations and systems

1.1 ★☆☆ Solution → p. 75

Which of the following equations is linear in the variables x, y, z?

A) $2^x + y = 3$

B) $(\ln 2)\,x - \sqrt{5}\,y = z + 1$

C) $xy - z = 0$

D) $x^{-1} + y + z = 4$

E) $\sqrt{x+y} = z$

1.2 ★☆☆ Solution → p. 75

a. Write the augmented matrix of the following system and determine whether the triples $(1,2,1)$ and $(5,0,0)$ are solutions of the system.

$$\begin{aligned} 3x - y + 2z &= 3 \\ x + 4z &= 5 \\ -2y + z &= -3 \end{aligned}$$

b. Write the system whose augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & 3 & 1 & 0 \end{array}\right]$ in the unknowns x_1, x_2, x_3.

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1.3 ★★☆ Solution → p. 75

The lines $d_1 : x + 2y = 4$, $d_2 : 3x - y = 5$, and $d_3 : ax + y = 3$ in the plane are given.

a. For what value of a do the three lines pass through a common point?

b. For which values of a is d_3 parallel to d_1 or to d_2?

c. For the other values of a, how many solutions does the system of three equations have? Interpret geometrically.

1.4 ★☆☆ Solution → p. 75

The sum of the digits of a three-digit number is 14. The hundreds digit equals the sum of the other two digits. When the order of the digits is reversed, the resulting number is 198 less than the original number. Set up a linear system and find the number.

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Gaussian elimination

1.5 ★☆☆ Solution → p. 76

Which of the following matrices is in reduced row echelon form?

A) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ **B)** $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **C)** $\begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **D)** $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **E)** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 0 \end{bmatrix}$

1.6 ★☆☆ Solution → p. 76

Solve the systems whose augmented matrices have been brought by row operations to the following forms. Identify the pivot columns and the free variables.

a. $\left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 1 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $(x_1, \ldots, x_4)$

b. $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 5 \end{array}\right]$ (x_1, x_2, x_3)

1.7 ★★☆ Solution → p. 76

Solve the following system by Gaussian elimination (reducing to row echelon form and back-substituting).

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + 3y + z &= 10 \\ -x + y + 4z &= 9 \end{aligned}$$

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1.5

← Problem p. 6

The conditions for reduced row echelon form: zero rows are at the bottom; the first nonzero entry of each row is 1 (the pivot); the pivot of a lower row is farther to the right; all entries in a pivot column other than the pivot are 0.

- (A): the pivot of the third row (column 2) is to the left of the pivot of the second row (column 3).
- (B): the first entry of the second row is 2, so the pivot is not 1.
- (C): it is in row echelon form, but there is a 5 above the pivot in column 4; it is not reduced.
- (D): all conditions hold. Column 3 has no pivot; the 3 and -1 there are unconstrained.
- (E): the zero row is at the top.

ANSWER D

1.6

← Problem p. 6

a. The pivots are in columns 1 and 3; x_2 and x_4 are free: $x_2 = s$, $x_4 = t$. The rows give

$$x_3 - x_4 = 4, \qquad x_1 - 2x_2 + 3x_4 = 1$$

Therefore

$$x_1 = 1 + 2s - 3t, \quad x_2 = s, \quad x_3 = 4 + t, \quad x_4 = t.$$

The zero row ($0 = 0$) carries no information. The system has infinitely many solutions.

b. The last row says $0x_1 + 0x_2 + 0x_3 = 5$; this can never hold. The system is inconsistent. **Common mistake:** back-substituting from the upper rows and writing a "solution"; first look for an inconsistent row.

ANSWER a. $(1 + 2s - 3t,\ s,\ 4 + t,\ t)$ b. no solution

1.7

← Problem p. 6

Start with the augmented matrix. $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 3 & 1 & 10 \\ -1 & 1 & 4 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & 4 \\ 0 & 3 & 3 & 12 \end{array}\right].$$

$R_3 \to R_3 + 3R_2$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 12 & 24 \end{array}\right].$$

Back-substitution: $12z = 24 \Rightarrow z = 2$; $-y + 6 = 4 \Rightarrow y = 2$; $x + 4 - 2 = 3 \Rightarrow x = 1$.

Check: $-1 + 2 + 8 = 9$ ✓ (third equation). The pivots are $1, -1, 12$; here the pivots need not be 1 since we did not scale the rows, but in reduced row echelon form they must be leading 1s.

ANSWER $(x, y, z) = (1, 2, 2)$

1.8

← Problem p. 7

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 4 \\ 0 & -1 & 1 & 1 & -3 \\ 0 & -2 & 2 & 1 & -4 \end{array}\right].$$

$R_3 \to R_3 - 2R_2$, then $R_3 \to -R_3$:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 4 \\ 0 & -1 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right].$$

Working upward: $R_2 \to R_2 - R_3$, $R_2 \to -R_2$, then $R_1 \to R_1 - R_2$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right].$$

The pivots are in columns 1, 2, and 4; $x_3 = t$ is free:

$$x_1 = 3 - 2t, \quad x_2 = 1 + t, \quad x_3 = t, \quad x_4 = -2.$$

Check ($t = 0$, $(3, 1, 0, -2)$): $3 + 1 = 4$, $6 + 1 - 2 = 5$, $3 - 1 - 2 = 0$ ✓. Note: x_4 equals -2 in every solution; the only free variable is x_3.

ANSWER $(3 - 2t,\ 1 + t,\ t,\ -2),\ t \in \mathbb{R}$

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1.9

← Problem p. 8

In a homogeneous system the right-hand side stays 0 throughout, so it suffices to reduce the coefficient matrix. $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$:

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(Second step: $R_3 \to R_3 - R_2$, $R_2 \to \frac{1}{3}R_2$, $R_1 \to R_1 + R_2$.) The pivots are in columns 1 and 3; the second column has *no* pivot. The free variables are $x_2 = s$, $x_4 = t$:

$$x_1 = -2s - 2t, \quad x_2 = s, \quad x_3 = -t, \quad x_4 = t.$$

For example, $s = 1$, $t = 0$: $(-2, 1, 0, 0)$. **Check** ($s = 0$, $t = 1$, $(-2, 0, -1, 1)$): $-2 + 1 + 1 = 0$, $-4 - 1 + 5 = 0$, $-2 - 2 + 4 = 0$ ✓.

ANSWER $(-2s - 2t,\ s,\ -t,\ t)$; for example $(-2, 1, 0, 0)$

1.10

← Problem p. 8

A homogeneous system is always consistent: $x_1 = \cdots = x_4 = 0$ is a solution, so (B) is false. Since there are at most 3 rows, the number of pivots is at most 3; $4 - 3 = 1$, so there is at least one free variable. A consistent system with a free variable has infinitely many solutions: (C) is true, (A) is false.

- (D): the number of free variables is $4 -$ (number of pivots); if the number of pivots is 1 or 2, there are 3 or 2 free variables.
- (E): there can be three pivots; then there is no zero row. Example: $x_1 = 0$, $x_2 = 0$, $x_3 + x_4 = 0$.

ANSWER C

1.11

← Problem p. 8

a. True. With $R_2 \to R_2 - R_1$ the second equation becomes $-2y + (k-1)z = 2$. Since the coefficient of y is $-2 \neq 0$, the second row has a pivot whatever k is; an inconsistent row of the form $[\,0\ 0\ 0 \mid c\,]$ cannot occur. The system is consistent for every k, and z is free:

$$x = 2 - \tfrac{k+1}{2}z, \qquad y = -1 + \tfrac{k-1}{2}z.$$

Check ($z = 0$, $(2, -1, 0)$): $2 - 1 + 0 = 1$, $2 + 1 + 0 = 3$ ✓. Note: the coefficient rows $(1, 1, 1)$ and $(1, -1, k)$ of the two equations are not proportional for any k; having more unknowns than equations does not by itself guarantee consistency.

b. True. Let an equation be $a_1x_1 + \cdots + a_nx_n = b$. Substituting **p** and **q** gives b in both cases; for their average the left side is $\frac{1}{2}b + \frac{1}{2}b = b$. The same holds for every equation.

c. False (if the system is not homogeneous). The same computation gives $b + b = 2b$; this fails if $b \neq 0$. Example: the solutions of $x + y = 1$ include $(1, 0)$ and $(0, 1)$, but $(1, 1)$ is not a solution. For homogeneous systems it is true.

d. False. The absence of a pivot in the last column only says that the system is *consistent*. Example: for the system $x + y = 2$, $2x + 2y = 4$, applying $R_2 \to R_2 - 2R_1$ gives the row echelon form $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$; there is no pivot in the last column, but y is free and the solution is $(2 - t,\ t)$, so there are infinitely many solutions. For a unique solution, every coefficient column must also have a pivot.

ANSWER a. T b. T c. F d. F

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