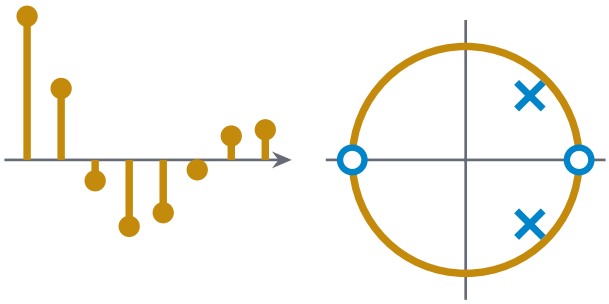
DIGITAL SIGNAL PROCESSING

PROBLEM BOOK

SAMPLE PAGES

8 chapters · 144 problems · step-by-step solutions



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How to use this book

- Each chapter opens with a reminder box summarizing the definitions and formulas the problems rely on.
- Difficulty: ★☆☆ basic ★★☆ intermediate ★★★ advanced. Each chapter has a few multiple-choice concept questions; its last problem is an exam-style mixed problem combining several topics.
- The dotted area below each problem is for your own work. Solve first; if stuck, go back to the reminder box and the lecture notes, and look at the solution last.
- All solutions are at the end of the book. The Solution → link next to a problem takes you to its solution, and ← Problem takes you back. Short answers are in the **ANSWER** box.
- **Shared chapters:** Chapters 7–9 (discrete-time Fourier transform, sampling, z-transform) are the same as in *Signals and Systems*. They are included here for schools whose second course begins with these topics; for others they serve as review. The book starts with Chapter 7: of the earlier chapters, only convolution and the continuous-time Fourier transform are assumed, and they are summarized in the reminder boxes where needed.
- **Midterm and final:** The midterm usually covers the discrete-time Fourier transform, sampling, the z-transform, and the discrete Fourier transform (Chapters 7–10); the final covers the FFT and spectral analysis, discrete-time structures and quantization, and IIR and FIR filter design (Chapters 11–14). The selection of topics varies from school to school; choose the chapters to study according to your own course syllabus.

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CHAPTER 7

DISCRETE-TIME FOURIER TRANSFORM

convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

DTFT $\quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

inverse DTFT $\quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$

shift, impulse $\quad \delta[n-n_0] \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega n_0}$

exponential ($|a|<1$) $\quad a^n u[n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{1-a\,e^{-j\omega}}$

rectangular

$$u[n]-u[n-L] \overset{\mathscr{F}}{\longleftrightarrow} \frac{\sin(\omega L/2)}{\sin(\omega/2)}\,e^{-j\omega(L-1)/2}$$

ideal lowpass $\quad \frac{\sin(\omega_c n)}{\pi n} \overset{\mathscr{F}}{\longleftrightarrow} \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$

complex exponential

$$e^{j\omega_0 n} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\sum_k \delta(\omega-\omega_0-2\pi k)$$

shift, modulation

$$x[n-n_0] \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega n_0}X, \quad e^{j\omega_0 n}x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

time reversal, conjugate

$$x[-n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{-j\omega}), \quad x \text{ real} : X(e^{-j\omega}) = X^*(e^{j\omega})$$

differentiation in frequency

$$n\,x[n] \overset{\mathscr{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

convolution $\quad x[n]*h[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega})H(e^{j\omega})$

multiplication

$$x[n]w[n] \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})\,d\theta$$

Parseval $\quad \sum_n |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\,d\omega$

response to a sinusoid

$$\cos(\omega_0 n) \to |H(e^{j\omega_0})|\cos\big(\omega_0 n + \angle H(e^{j\omega_0})\big)$$

group delay $\quad \tau(\omega) = -\frac{d}{d\omega}\angle H(e^{j\omega})$

In this chapter systems are linear and time-invariant (LTI): a system is completely determined by its impulse response $h[n]$, and the output is the convolution of the input with the impulse response, $y = x * h$. The frequency response $H(e^{j\omega})$ of the system is the discrete-time Fourier transform (DTFT) of $h[n]$. The frequency variable ω is the discrete-time frequency (in rad/sample); the DTFT is periodic with period 2π, so only the interval $-\pi < \omega \le \pi$ is plotted, and the highest discrete-time frequency is $\omega = \pi$. Finite sequences are written inside $\{\dots\}$, and the index of the first element is stated separately; samples not shown in stem plots are zero. Systems given by difference equations are causal and initially at rest; in block diagrams D is a one-sample delay. Stability is meant in the bounded-input bounded-output (BIBO) sense: $\sum|h[n]| < \infty$. In the figures the given signal and its spectrum are drawn in blue; the signal and spectrum found, the impulse response and the frequency response in red (a second response in the same figure in dashed red); the ideal (target) response in dashed gray.

The DTFT and basic pairs

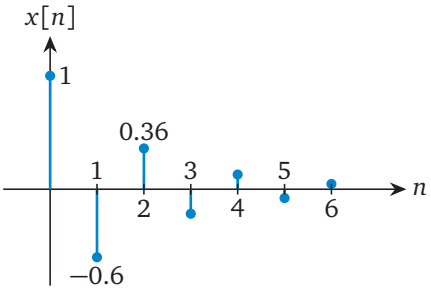
7.1 ★☆☆ Solution →

Which of the following discrete-time signals has the highest frequency (oscillates fastest from sample to sample)?

A) $\cos(0.9\pi n)$ **B)** $\cos(1.6\pi n)$ **C)** $\cos(1.9\pi n)$ **D)** $\cos(2.2\pi n)$ **E)** $\cos(3.9\pi n)$

7.3 ★☆☆ Solution →

The signal in the figure is $x[n] = (-0.6)^n u[n]$.



a. Find $X(e^{j\omega})$.

b. Find $|X(e^{j\omega})|$ at $\omega = 0$, $\pi/2$ and π; find the phase at $\omega = \pi/2$. At which frequencies is the energy of the signal concentrated? Why?

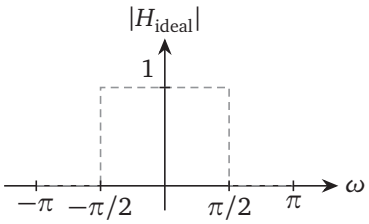
c. Find the energy of the signal.

Discrete-time filters, phase and group delay

7.16 ★★★ Solution →

A causal filter is obtained by delaying the impulse response of the ideal lowpass filter with cutoff frequency $\pi/2$ by 4 samples and truncating it to the interval $0 \le n \le 8$:

$$h[n] = \frac{\sin\big(0.5\pi(n-4)\big)}{\pi(n-4)}, \quad 0 \le n \le 8.$$



The figure shows the ideal magnitude response to be approximated.

a. Find $h[0], \ldots, h[8]$. What kind of symmetry does the sequence have?

b. Write $H(e^{j\omega})$ in the form $e^{-j4\omega}A(\omega)$ (A real). What is the group delay?

c. Find $A(0)$, $A(\pi/2)$ and $A(\pi)$; compare them with the ideal values.

d. Find the output for the input $x[n] = \cos(0.2\pi n) + \cos(0.8\pi n)$.

CHAPTER 10
DISCRETE FOURIER TRANSFORM

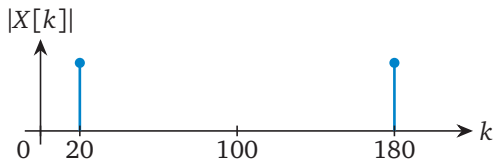
DFT	$X[k]=\sum_{n=0}^{N-1}x[n]W_N^{kn}$	circular shift	$x[((n-m))_N] \leftrightarrow W_N^{km}X[k]$
inverse DFT	$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn}$	frequency shift	$W_N^{-ln}x[n] \leftrightarrow X[((k-l))_N]$
phase factor	$W_N=e^{-j2\pi/N},\quad W_N^N=1$	real $x[n]$	$X[((-k))_N]=X^*[k]$
DTFT	$X(e^{j\omega})=\sum_n x[n]e^{-j\omega n}$	Parseval	$\sum_{n=0}^{N-1}\lvert x[n]\rvert^2=\frac{1}{N}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2$
DTFT samples	$X[k]=X(e^{j\omega})\big\rvert_{\omega=2\pi k/N}$	circular convolution	$x_1 \circledast_N x_2 \leftrightarrow X_1X_2$
DFT bin	$\omega_k=\frac{2\pi k}{N},\quad f_k=\frac{kf_s}{N}$	(definition of $\circledast_N$)	$\sum_{m=0}^{N-1}x_1[m]\,x_2[((n-m))_N]$
		time aliasing	$y_N[n]=\sum_r y[n+rN]$
		linear = circular	$N \geq L+M-1$
		convolution sum	$y[n]=\sum_k x[k]h[n-k]$

In this chapter the DFT is N-point: sequences are defined for $0 \leq n \leq N-1$ and DFT values for $0 \leq k \leq N-1$. Each k is a DFT bin at the frequency $f_k = kf_s/N$; the bin spacing is $\Delta f = f_s/N$. Finite sequences are written inside $\{\ldots\}$, starting from $n=0$ unless stated otherwise; samples not shown in stem plots are zero. $((n))_N$ is the remainder of n divided by N ($((-1))_8 = 7$). $x_1[n] \circledast_N x_2[n]$ is the N-point circular convolution, and $x[n] * h[n]$ is the linear convolution. L and M are the lengths of the two sequences being convolved. Circular convolution is the linear result with copies shifted by multiples of N added on top of one another; this is called time aliasing. Frequency in hertz is denoted by f and the sampling frequency by f_s; the discrete-time angular frequency is $\omega = 2\pi f/f_s$ (in rad/sample). In the figures the input is drawn in blue and the impulse response in red.

Definition of the DFT and its relation to the DTFT

10.1 ★☆☆ Solution →

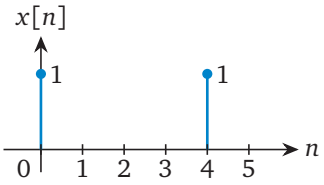
The signal of a microphone recording the hum of a distribution transformer is passed through an anti-aliasing filter that cuts off everything above 500 Hz and is then sampled at $f_s = 1$ kHz, and the $N = 200$-point DFT of the recording is computed. For $0 \leq k \leq 199$, $|X[k]|$ has a significant value only at $k = 20$ and $k = 180$ (figure); at all other k it is negligibly small. What is the frequency of the hum?



A) 20 Hz **B)** 100 and 900 Hz **C)** 100 Hz **D)** 900 Hz **E)** 50 Hz

10.3 ★★☆ Solution →

The sequence $x[n] = \delta[n] + \delta[n-4]$ shown in the figure is given.



a. Find $X(e^{j\omega})$, and show that $|X(e^{j\omega})| = 2|\cos 2\omega|$. What are its zeros in $0 \le \omega < 2\pi$?

b. Compute the 5-point DFT $X_5[k]$ from the definition, and show that the values $|X_5[k]|$ are samples of $|X(e^{j\omega})|$.

c. $x[n]$ is zero-padded and its N-point DFT is computed ($N \ge 5$). What is the smallest N for which one of the DFT bins falls on a zero of $|X(e^{j\omega})|$? Write $X[k]$ for this N.

Mixed problems

10.16 ★★★ Solution →

Audio software processes the real left and right channels of a stereo recording, $l[n]$ and $r[n]$, with a single complex DFT: $z[n] = l[n] + j\,r[n]$. For a 4-sample segment the result is

$$Z[k] = \{2 + 6j,\ 2,\ 4 - 4j,\ 2j\}.$$

a. Show that the following relations hold:

$$L[k] = \frac{Z[k] + Z^*[((-k))_4]}{2}, \qquad R[k] = \frac{Z[k] - Z^*[((-k))_4]}{2j}.$$

b. Find $L[k]$ and $R[k]$.

c. Find $l[n]$ and $r[n]$; check the result with $z[0]$.

CHAPTER 13

IIR FILTER DESIGN

frequencies	$\omega = \Omega T = 2\pi f / f_s$
ripple	$1 - \delta_p = 10^{-A_p/20}$
attenuation	$\delta_s = 10^{-A_s/20}$
Laplace pair	$e^{s_k t} u(t) \overset{\mathscr{L}}{\longleftrightarrow} \dfrac{1}{s - s_k}$
frequency response	$H_c(j\Omega) = H_c(s)\big\|_{s=j\Omega}$
stability	$\mathrm{Re}\{s_k\} < 0$ (causal)
Butterworth	$\lvert H_c(j\Omega)\rvert^2 = \dfrac{1}{1 + (\Omega/\Omega_c)^{2N}}$
order	$N \geq \dfrac{\log_{10} D}{2\log_{10}(\Omega_s/\Omega_p)}$
	$D = \dfrac{10^{A_s/10} - 1}{10^{A_p/10} - 1}$
poles	$s_k = \Omega_c \, e^{j\pi(2k+N-1)/(2N)}$
impulse invariance	$h[n] = T\, h_c(nT)$
	$\dfrac{A_k}{s - s_k} \rightarrow \dfrac{TA_k}{1 - e^{s_k T} z^{-1}}$
aliasing	$H(e^{j\omega}) = \sum_k H_c\Big(j\,\dfrac{\omega - 2\pi k}{T}\Big)$
(if $h_c(0^+) \neq 0$)	$+\, T\, h_c(0^+)/2$
bilinear transformation	$s = \dfrac{2}{T} \cdot \dfrac{1 - z^{-1}}{1 + z^{-1}}$
	$z = \dfrac{1 + sT/2}{1 - sT/2}$
frequency warping	$\Omega = \dfrac{2}{T} \tan \dfrac{\omega}{2}$
group delay	$\tau(\omega) = -\dfrac{d}{d\omega} \angle H(e^{j\omega})$
pole pair $re^{\pm j\omega_0}$	$\Delta\omega \approx 2(1 - r)$ $(r \approx 1)$

In this chapter IIR (infinite impulse response) digital filters are designed from analog prototypes; the analog (continuous-time) and digital (discrete-time) domains are used together: Ω is the analog angular frequency in rad/s, and ω is the digital frequency in rad/sample; f is the frequency in hertz, f_s the sampling frequency, and $T = 1/f_s$. The analog filter is denoted $H_c(s)$ and the digital filter $H(z)$. $s = \sigma + j\Omega$ is the complex variable of the Laplace transform; a causal analog filter is stable if all of its poles lie in the left half of the s-plane ($\mathrm{Re}\{s_k\} < 0$). In the formula for s_k, $k = 1, \ldots, N$. The passband ripple A_p and the stopband attenuation A_s are positive numbers in decibels: $20\log_{10}|H| \geq -A_p$ in the passband and $20\log_{10}|H| \leq -A_s$ in the stopband; the maximum gain is 1. Impulse invariance uses $h[n] = T\, h_c(nT)$; the factor T approximately preserves the gain; the overlap of spectral replicas (aliasing) causes small differences. In the figures the given signal is drawn in blue, the impulse response and the frequency response in red, and the ideal response in dashed gray; gray shading marks only the forbidden regions in tolerance schemes.

Design specifications and analog prototypes

13.1 ★☆☆ Solution →

Which of the following is true for an Nth-order analog Butterworth lowpass filter?

A) At the cutoff frequency $|H_c(j\Omega_c)| = 1/\sqrt{2}$ (−3 dB); this holds for every N, and the magnitude decreases with increasing Ω without ever rising.

B) In the passband the magnitude ripples equally between 1 and $1/\sqrt{1+\varepsilon^2}$.

C) The poles lie on the circle of radius Ω_c; if N is odd, one of the poles is at $s = 0$.

D) The attenuation at the cutoff frequency increases with the order; for $N = 4$ it is 6 dB.

E) In the stopband the magnitude decreases by 20 dB per decade, independently of N.

Impulse invariance

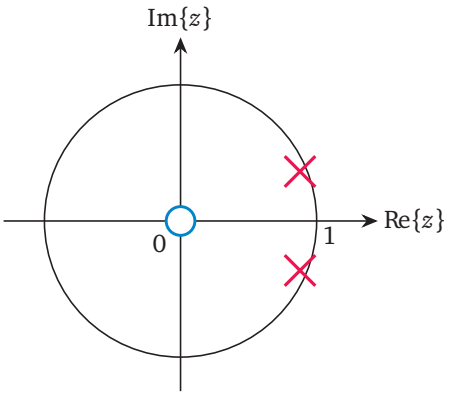
13.8

★★☆ Solution →

The digital filter in the accelerometer channel of a vehicle navigation system was obtained from an analog prototype by impulse invariance at $f_s = 200$ Hz:

$$H(z) = \frac{0.1452\,z^{-1}}{1 - 1.7554\,z^{-1} + 0.9025\,z^{-2}}$$

(the poles and the zero at $z = 0$ are shown in the figure). The prototype has the form $H_c(s) = K/[(s+\alpha)^2 + \beta^2]$ $(\alpha, \beta > 0)$.



a. Find the radius and the angle of the poles, and α and β.

b. Find K; compare the analog and digital DC gains.

c. If $\beta + 2\pi/T$ were taken instead of β, what would the poles in the z-plane be? How is this ambiguity resolved?

The bilinear transformation and frequency prewarping

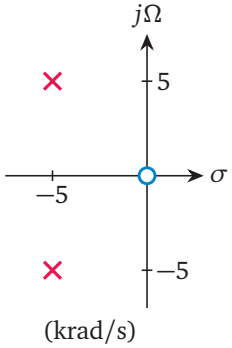
13.12

★★☆ Solution →

The analog bandpass filter

$$H_c(s) = \frac{10^4 s}{s^2 + 10^4 s + 5 \cdot 10^7}$$

(s in rad/s), whose poles and finite zero are shown in the figure, is digitized by the bilinear transformation at $f_s = 5$ kHz (without prewarping).



a. Map the poles and the zero with $z = (1 + sT/2)/(1 - sT/2)$. Is the digital filter stable? Where does the zero at $s = \infty$ go?

b. Find $H(z)$.

c. The analog peak is at $\Omega_0 = \sqrt{5 \cdot 10^7}$ rad/s. At what frequency (in Hz) is the digital peak? Does the peak gain change?

SOLUTIONS

7 DISCRETE-TIME FOURIER TRANSFORM

7.1

← Problem

A discrete-time sinusoid is periodic in ω with period 2π: $\cos((\omega_0+2\pi k)n)=\cos(\omega_0 n)$, because n is an integer. Also $\cos(-\omega_0 n)=\cos(\omega_0 n)$. Multiples of 2π are first subtracted from each frequency, and then, if necessary, the minus sign is dropped using $\cos(-\omega_0 n)=\cos(\omega_0 n)$; this brings the frequency into the interval $0\le\omega\le\pi$:

ω_0	0.9π	1.6π	1.9π	2.2π	3.9π
equivalent	0.9π	0.4π	0.1π	0.2π	0.1π

In discrete time the highest frequency is $\omega=\pi$ ($(-1)^n$); the closest to π is 0.9π.

- (B): $1.6\pi=2\pi-0.4\pi$; the signal is identical to $\cos(0.4\pi n)$.
- (C) and (E): Both are $\cos(0.1\pi n)$; unlike in continuous time, a large ω_0 does not mean fast oscillation in discrete time.
- (D): $2.2\pi-2\pi=0.2\pi$.

ANSWER A

7.3

← Problem

a. $a=-0.6$, $|a|<1$: the sequence is absolutely summable, and the DTFT exists:

$$X(e^{j\omega})=\sum_{n=0}^{\infty}(-0.6\,e^{-j\omega})^n=\frac{1}{1+0.6\,e^{-j\omega}}.$$

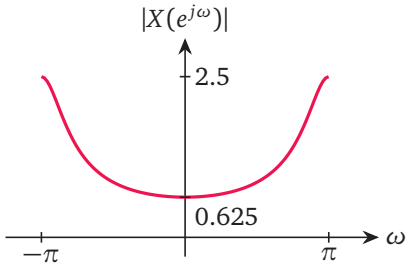
b. $|1+0.6e^{-j\omega}|^2=1.36+1.2\cos\omega$:

$$|X(e^{j0})|=\frac{1}{1.6}=0.625,\qquad |X(e^{j\pi})|=\frac{1}{0.4}=2.5,$$

$$X(e^{j\pi/2})=\frac{1}{1-0.6j},\quad |X|=\frac{1}{\sqrt{1.36}}=0.8575,$$

$$\angle X(e^{j\pi/2})=\arctan 0.6=0.5404\text{ rad }(30.96^\circ).$$

The magnitude increases toward $\omega=\pi$ (figure): the energy is at high frequencies. The sign of the samples alternates at every step; $(-0.6)^n=0.6^n(-1)^n$, that is, the envelope 0.6^n is multiplied by $(-1)^n$, the sequence at $\omega=\pi$.



c. Energy $\sum|x[n]|^2=\sum 0.36^n$:

$$E=\frac{1}{1-0.36}=1.5625.$$

ANSWER a. $X(e^{j\omega})=1/(1+0.6e^{-j\omega})$ b. 0.625; 0.8575; 2.5; phase 0.5404 rad; energy at high frequencies ($\omega\approx\pi$) c. $E=1.5625$

7.16

← Problem

a. With $k=n-4$, $h=\sin(0.5\pi k)/(\pi k)$; the limit at $k=0$ is 0.5:

k	0	±1	±2	±3	±4
h	0.5	$1/\pi$	0	$-1/(3\pi)$	0

$1/\pi=0.3183$, $1/(3\pi)=0.1061$; for $n=0,\dots,8$:

$$h=\{0,-0.1061,0,0.3183,0.5,0.3183,0,-0.1061,0\}.$$

The sequence is symmetric about $n=4$: $h[n]=h[8-n]$ (e.g. $h[1]=h[7]=-0.1061$).

b. The symmetric pairs are combined by factoring out $e^{-j4\omega}$: the pair $h[4\pm k]$ gives $2h\cos(k\omega)$.

$$A(\omega)=0.5+\frac{2}{\pi}\cos\omega-\frac{2}{3\pi}\cos 3\omega.$$

The phase is -4ω (linear; with jumps of π where $A<0$), and the group delay is $\tau=4$ samples at every frequency: all components pass with the same delay, and there is no phase distortion. (Since $A(\omega)$ varies with frequency, there is magnitude distortion.)

c. $2/\pi=0.6366$, $2/(3\pi)=0.2122$:

$$A(0)=0.5+0.6366-0.2122=0.9244,$$

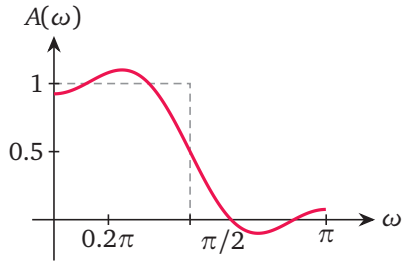
$$A(\pi/2)=0.5,$$

$$A(\pi)=0.5-0.6366+0.2122=0.0756.$$

Ideally the value is 1 in the passband and 0 in the

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stopband. Truncation leaves ripple in the passband and leakage in the stopband (figure); at the cutoff frequency the gain is one half (−6 dB).



d. $\cos(0.2\pi) = 0.8090$, $\cos(0.6\pi) = -0.3090$, $\cos(0.8\pi) = -0.8090$, $\cos(2.4\pi) = 0.3090$:

$$A(0.2\pi) = 0.5 + 0.6366(0.8090) - 0.2122(-0.3090) = 1.0806,$$

$$A(0.8\pi) = 0.5 - 0.6366(0.8090) - 0.2122(0.3090) = -0.0806.$$

$$y[n] = 1.0806\cos\big(0.2\pi(n-4)\big) - 0.0806\cos\big(0.8\pi(n-4)\big).$$

The passband component is boosted by 8%, and the stopband component is reduced to 8% (with its sign reversed); both are delayed by 4 samples.

ANSWER a. $h[4] = 0.5$, $h[3] = h[5] = 0.3183$, $h[1] = h[7] = -0.1061$, other samples 0; $h[n] = h[8-n]$ b. $A(\omega) = 0.5 + (2/\pi)\cos\omega - (2/(3\pi))\cos 3\omega$; $\tau = 4$ c. $A(0) = 0.9244$, $A(\pi/2) = 0.5$, $A(\pi) = 0.0756$ d. $y[n] = 1.0806\cos(0.2\pi(n-4)) - 0.0806\cos(0.8\pi(n-4))$

10 DISCRETE FOURIER TRANSFORM

10.1

← Problem

The DFT bin spacing is $\Delta f = f_s/N = 1000/200 = 5$ Hz; bin k corresponds to the frequency $f_k = kf_s/N$:

$$f_{20} = 20(5) = 100 \text{ Hz}, \qquad f_{180} = 180(5) = 900 \text{ Hz}.$$

But since the anti-aliasing filter cuts off everything above 500 Hz, the recording cannot contain a 900 Hz component. For a real sequence $X[N-k] = X^*[k]$, so bin $k = 180 = 200 - 20$ is the conjugate-symmetric counterpart of $k = 20$: it corresponds to the $e^{-j\omega n}$ half of the cosine, that is, to −100 Hz ($(180-200)\cdot 5$). The recording contains only one component: 100 Hz (twice the mains frequency; vibration of the transformer core).

- (A): The index k was taken directly as hertz.
- (B): $k = 180$ was counted as a separate component; the conjugate symmetry was forgotten.
- (D): Only the conjugate (mirror) bin was taken; 900 Hz lies in the band cut off by the filter.
- (E): $f = (k/N)(f_s/2)$ was used; the N bins of the DFT span 0 to f_s (not $f_s/2$).

ANSWER C

10.3

← Problem

a. The DTFT of a shifted impulse is $e^{-j\omega n_0}$:

$$X(e^{j\omega}) = 1 + e^{-j4\omega} = e^{-j2\omega}\big(e^{j2\omega} + e^{-j2\omega}\big) = 2\cos(2\omega)\,e^{-j2\omega}.$$

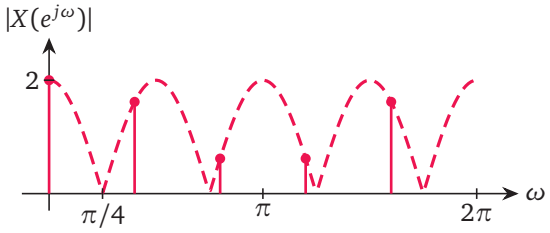
Since $|e^{-j2\omega}| = 1$, $|X(e^{j\omega})| = 2|\cos 2\omega|$. The zeros are where $\cos 2\omega = 0$, that is, at $2\omega = \pi/2 + m\pi$:

$$\omega = \pi/4, \quad 3\pi/4, \quad 5\pi/4, \quad 7\pi/4.$$

b. $X_5[k] = \sum_n x[n]\,W_5^{kn} = 1 + W_5^{4k} = 1 + e^{j2\pi k/5}$ ($W_5^{5k} = 1$). With $|1 + e^{j\theta}| = 2|\cos(\theta/2)|$:

$$|X_5[k]| = 2\left|\cos\frac{\pi k}{5}\right|: \quad 2,\ 1.618,\ 0.618,\ 0.618,\ 1.618.$$

The value of the DTFT at $\omega_k = 2\pi k/5$ is $2|\cos(4\pi k/5)|$; for $k = 1$ it is $2|\cos 144°| = 1.618$, for $k = 2$ it is $2|\cos 288°| = 0.618$: the same numbers. Since $N \geq 5$ (the length of the sequence), the DFT values are exact samples of the DTFT (figure).



The stems are the magnitudes of the 5-point DFT; all of them lie on the dashed DTFT curve.

c. DFT bin k is at $\omega_k = 2\pi k/N$; for it to fall on a zero, $2\pi k/N = (2m+1)\pi/4$, that is, $k/N = (2m+1)/8$.

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Since the numerator is odd, this fraction cannot be reduced; N must be a multiple of 8. The smallest is $N = 8$ ($k = 1$: $\omega = \pi/4$). $X_8[k] = 1 + W_8^{4k} = 1 + (-1)^k$:

$$X_8[k] = \{2, 0, 2, 0, 2, 0, 2, 0\}.$$

The odd-indexed bins $(\pi/4, 3\pi/4, \ldots)$ fall exactly on the zeros.

ANSWER a. $X(e^{j\omega}) = 2\cos(2\omega)e^{-j2\omega}$; zeros $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$ b. $X_5[k] = 1 + e^{j2\pi k/5}$; $|X_5[k]| = 2, 1.618, 0.618, 0.618, 1.618$ c. $N = 8$; $X_8[k] = \{2, 0, 2, 0, 2, 0, 2, 0\}$

10.16

← Problem

a. $l = (z + z^*)/2$ and $r = (z - z^*)/(2j)$. The DFT of $z^*[n]$:

$$\sum_n z^*[n] W_N^{kn} = \Big(\sum_n z[n] W_N^{-kn}\Big)^* = Z^*[((-k))_N].$$

The two relations follow by linearity. $(((-k))_4$: $0, 3, 2, 1$ for $k = 0, 1, 2, 3$.)

b. $Z^*[((-k))_4] = \{2 - 6j,\ -2j,\ 4 + 4j,\ 2\}$:

$$L[k] = \tfrac{1}{2}\{4,\ 2 - 2j,\ 8,\ 2 + 2j\} = \{2,\ 1 - j,\ 4,\ 1 + j\},$$

$$R[k] = \tfrac{1}{2j}\{12j,\ 2 + 2j,\ -8j,\ -2 + 2j\}$$
$$= \{6,\ 1 - j,\ -4,\ 1 + j\}.$$

Both have the form of the DFT of a real sequence: $k = 0, 2$ are real, and $k = 1, 3$ are complex conjugates.

c. In the inverse DFT $W_4^{-kn} = j^{kn}$; since the terms $1 - j$ and $1 + j$ are complex conjugates, their sum is $2\,\mathrm{Re}\{(1 - j)j^n\}$: $2, 2, -2, -2$ for $n = 0, 1, 2, 3$.

$$l[n] = \tfrac{1}{4}\big[2 + 4(-1)^n + 2\,\mathrm{Re}\{(1 - j)j^n\}\big],$$

$$l[n] = \tfrac{1}{4}\{8, 0, 4, -4\} = \{2, 0, 1, -1\},$$

$$r[n] = \tfrac{1}{4}\big[6 - 4(-1)^n + 2\,\mathrm{Re}\{(1 - j)j^n\}\big],$$

$$r[n] = \tfrac{1}{4}\{4, 12, 0, 8\} = \{1, 3, 0, 2\}.$$

Check: $z[0] = l[0] + j\,r[0] = 2 + j$; from the inverse DFT, $z[0] = \frac{1}{4}\sum Z[k] = \frac{1}{4}(8 + 4j) = 2 + j$.

The DFTs of two real sequences are obtained with one complex DFT and a few additions; the computational load is roughly halved.

ANSWER a. $z^*[n] \leftrightarrow Z^*[((-k))_4]$ b. $L[k] = \{2, 1 - j, 4, 1 + j\}$, $R[k] = \{6, 1 - j, -4, 1 + j\}$ c. $l[n] = \{2, 0, 1, -1\}$, $r[n] = \{1, 3, 0, 2\}$; $z[0] = 2 + j$

13 IIR FILTER DESIGN

13.1

← Problem

At $\Omega = \Omega_c$, $|H_c|^2 = 1/(1 + 1^{2N}) = 1/2$; whatever N is, $20\log_{10}(1/\sqrt{2}) = -3.01$ dB. Since $(\Omega/\Omega_c)^{2N}$ is increasing for $\Omega > 0$, $|H_c|$ decreases monotonically: there is no ripple in the passband, and the response is maximally flat at $\Omega = 0$.

- (B): Equiripple behavior is a property of the Chebyshev I filter.
- (C): If N is odd, the angle of pole $k = (N + 1)/2$ is π: the pole is at $s = -\Omega_c$. A pole at $s = 0$ would lie on the $j\Omega$-axis; the filter would not be stable.
- (D): The value at the cutoff is 3.01 dB for every N; the order changes only the steepness of the transition.
- (E): For $\Omega \gg \Omega_c$, $|H_c| \approx (\Omega_c/\Omega)^N$; the slope is $20N$ dB per decade.

ANSWER A

13.8

← Problem

a. $h_c(t) = (K/\beta)e^{-\alpha t}\sin(\beta t)u(t)$ (poles $-\alpha \pm j\beta$). After sampling, $h[n] = T(K/\beta)\,r^n \sin(\theta n)u[n]$, $r = e^{-\alpha T}$, $\theta = \beta T$; the denominator of the z-transform is $1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}$:

$$r^2 = 0.9025 \ \Rightarrow\ r = 0.95,$$

$$\cos\theta = \frac{1.7554}{2(0.95)} = 0.9239 \ \Rightarrow\ \theta = 0.3927 = \pi/8.$$

$T = 5$ ms:

$$\alpha = \frac{-\ln 0.95}{T} = \frac{0.05129}{0.005} = 10.26\ \mathrm{s}^{-1},$$

$$\beta = \frac{\theta}{T} = \frac{0.3927}{0.005} = 78.54\ \mathrm{rad/s}\ (12.5\ \mathrm{Hz}).$$

b. The numerator is $T(K/\beta)\,r\sin\theta = 0.1452$;

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$\sin(\pi/8) = 0.3827$:

$$K = \frac{0.1452 \cdot 78.54}{0.005 \cdot 0.95 \cdot 0.3827} = \frac{11.404}{0.0018177} = 6274.$$

$$H_c(0) = \frac{K}{\alpha^2 + \beta^2} = \frac{6274}{105.3 + 6168.5} = 1.000,$$

$$H(e^{j0}) = \frac{0.1452}{1 - 1.7554 + 0.9025} = \frac{0.1452}{0.1471} = 0.987.$$

The difference (−0.11 dB) is small: the pole frequency is 12.5 Hz and the Nyquist frequency 100 Hz; there is little aliasing.

c. $\beta' = 78.54 + 2\pi(200) = 1335.2$ rad/s (212.5 Hz): $\theta' = \beta' T = \pi/8 + 2\pi$, and the poles stay the same. e^{sT} is periodic in the imaginary part of s with period $2\pi/T$; every horizontal strip of the s-plane of width $2\pi/T$ in the direction of the imaginary axis is mapped onto the entire z-plane. The ambiguity is resolved by assuming that the analog poles lie in the strip $|\beta| < \pi/T = 628.3$ rad/s; this is the "bandlimited prototype" assumption that impulse invariance requires anyway.

ANSWER a. $r = 0.95$, $\theta = \pi/8$; $\alpha = 10.26\ \text{s}^{-1}$, $\beta = 78.54$ rad/s b. $K = 6274$; analog 1.000, digital 0.987 c. the poles do not change ($e^{j2\pi} = 1$); $|\beta| < \pi/T$ is assumed

13.12

← Problem

a. $T/2 = 10^{-4}$ s; the poles are $s = -5000 \pm j5000$, that is, $sT/2 = -0.5 \pm j0.5$:

$$z = \frac{0.5 + j0.5}{1.5 - j0.5} = \frac{(0.5 + j0.5)(1.5 + j0.5)}{2.5} = 0.2 + j0.4,$$

and its conjugate is $0.2 - j0.4$;
$|z| = \sqrt{0.2} = 0.4472 < 1$: the filter is stable. In general, if $\text{Re}\{s\} < 0$, then $|1 + sT/2| < |1 - sT/2|$, that is, $|z| < 1$: the left half-plane is mapped into the unit circle. The zero $s = 0 \to z = 1$; $s = \infty \to z = -1$.

b. $2/T = 10^4$; $s = 10^4 u$, $u = (1 - z^{-1})/(1 + z^{-1})$:

$$H = \frac{10^8 u}{10^8 u^2 + 10^8 u + 5 \cdot 10^7} = \frac{u}{u^2 + u + 0.5}.$$

The numerator and the denominator are multiplied by $(1 + z^{-1})^2$:

$$H(z) = \frac{1 - z^{-2}}{2.5 - z^{-1} + 0.5z^{-2}} = \frac{0.4(1 - z^{-2})}{1 - 0.4z^{-1} + 0.2z^{-2}}.$$

The roots of the denominator, $z^2 - 0.4z + 0.2 = 0$: $0.2 \pm j0.4$, the same as in part a.

c. $\Omega_0 = 7071$ rad/s (1125 Hz). The digital peak is at $\omega_0 = 2\arctan(\Omega_0 T/2)$:

$$\omega_0 = 2\arctan(0.7071) = 1.2310 \text{ rad},$$

$$f_0 = \frac{1.2310 \cdot 5000}{2\pi} = 979.6 \text{ Hz}.$$

Since $H(e^{j\omega}) = H_c\big(j(2/T)\tan(\omega/2)\big)$, the values of the response are preserved and only the frequency axis is warped: the peak gain $H_c(j\Omega_0) = 1$ remains unchanged, and its location moves from 1125 Hz to 979.6 Hz.

ANSWER a. poles $0.2 \pm j0.4$ ($|z| = 0.4472$, stable); zeros $z = 1$ and $z = -1$
b. $H(z) = 0.4(1 - z^{-2})/(1 - 0.4z^{-1} + 0.2z^{-2})$
c. 979.6 Hz (analog 1125 Hz); the peak gain remains 1

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